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Abstract

摘要

In this article, we study two related models of quantum geometry: generic random trees and two-dimensional causal triangulations. The Hausdorff and spectral dimensions that arise in these models are calculated, and their relationship with the structure of the underlying random geometry is explored. Modifications due to interactions with matter fields are also briefly discussed. The approach to the subject is that of classical statistical mechanics, and most of the tools come from probability and graph theory.

本文研究两个相关的量子几何模型: 一般随机树与二维因果三角剖分。我们计算了这些模型的豪斯多夫维数与谱维数，探究了它们与基础随机几何结构的关系，还简要讨论了物质场相互作用带来的模型修正。本文采用经典统计力学的研究方法，所用工具大多来自概率论与图论。

Keywords

关键词

Causal triangulation · Graph theoretic · Tree correspondence · Generic trees · Spectral dimension - Hausdorff dimension - Scaling amplitude

因果三角剖分 · 图论 · 树对应 · 泛树 · 谱维度 - 豪斯多夫维度 - 标度振幅

Introduction

引言

In this contribution, we adopt a graph theoretic and probabilistic perspective on two-dimensional causal dynamical triangulations (CDTs). We consider causal triangulations (CTs) as a particular class of planar graphs (defined in section ”Definition”) that are distributed according to a probabilistic law. Our primary goal is then to apply tools from graph and probability theory to analyze the large-scale geometric properties of these models. In particular, we exploit, in a variety of contexts, the correspondence (established in section ”Bijection Between CTs and Planar Trees”) between CTs and planar tree graphs (defined in section ”Random Trees”). In the process, we will define and study the generic random tree model, both because of its relation to branching processes and CTs and because of its independent interest as a testing ground for investigating various aspects of random graph models in general.

在本文中，我们从图论和概率论的视角研究二维因果动力学三角剖分 (CDT)。我们将因果三角剖分 (CT) 视作一类特殊的平面图 (在「定义」一节中给出定义)，其分布遵循某一概率规律。我们的核心目标是运用图论与概率论工具分析这些模型的大尺度几何性质。我们尤其在多种场景下利用了 CT 与平面树图 (在「随机树」一节中给出定义) 之间的对应关系，该对应关系在「CT 与平面树的双射」一节中建立。在此过程中，我们将定义并研究通用随机树模型，这既是因为它与分支过程和 CT 存在关联，也是因为它本身可作为一个独立的研究对象，是探究一般随机图模型各类性质的试验场。

We will consider two different ensembles of CTs. The first, much studied in the literature, is the grand canonical ensemble (defined in section ”Grand Canonical Ensemble and the Scaling Limit”) which is based on an expansion in the size of finite graphs. We use the planar tree correspondence to give an alternative account of the scaling behavior of loop correlation functions in the vicinity of the radius of convergence and determine the associated scaling Hausdorff dimension d_H . The second ensemble, to date less studied, is based on infinite CTs and can be thought of as an infinite volume limit suitable for studying local geometric characteristics of typical CTs. Again, we use the planar tree correspondence to investigate, in sections ”Hausdorff Dimension” and ”Spectral Dimension”, the fractal properties of CTs; in particular, we determine the local Hausdorff dimension d_h as well as the spectral dimension d_s (defined in section ”Definition of Spectral Dimension of Recurrent Graphs”). Indeed, for CTs, all three dimension exponents, d_H, d_h , and d_s have the value 2 . That this is not generally the case is illustrated by the generic trees for which $d_h = d_H = 2$, but whose spectral dimension is $d_s = \frac{4}{3}$ as shown in section ”Spectral Dimension of Generic Trees.”

我们将研究两种不同的 CT 系综。第一种系综已有大量文献研究，即巨正则系综 (在「巨正则系综与标度极限」一节中给出定义)，它建立在对有限图规模展开的基础上。我们利用平面树对应关系，对收敛半径附近的回路关联函数标度行为给出了另一种解释，并确定了对应的标度豪斯多夫维数 d_H 。第二种系综目前研究较少，它建立在无穷 CT 的基础上，可视为适用于研究典型 CT 局部几何特征的无穷体积极限。我们同样利用平面树对应关系，在「豪斯多夫维数」和「谱维数」两节中研究了 CT 的分形性质；具体而言，我们确定了局部豪斯多夫维数 d_h 以及谱维数 d_s (在「递归图谱维数的定义」一节中给出定义)。实际上，对于 CT，三个维数指数，d_H, d_h 以及 d_s 的取值均为 2。通用树的情况并非如此，正如「通用树的谱维数」一节所示，通用树满足 $d_h = d_H = 2$ ，但其谱维数为 $d_s = \frac{4}{3}$ ，这说明上述性质并不具有普遍性。

It is obviously of interest to understand the extent to which the results outlined above are universal and to investigate how they extend to the more general case of CTs coupled to statistical mechanical systems such as dimers and Ising spins. A few remarks on the rather sparse existing analytical results in this direction are collected in section ”Curvature and Matter Fields on the CDT.” Finally in section ”Where Next?,” we draw together the possible avenues for future research.

显然，探究上述结果的普适性范围，以及它们如何推广到 CT 与统计力学系统 (比如二聚物和伊辛自旋) 耦合的更一般情形，是十分有意义的。我们在「CDT 上的曲率与物质场」一节整理了现有关于该方向的稀少解析结果，并做了简要讨论。最后在「下一步研究方向」一节，我们总结了未来可开展研究的方向。

Preliminary on Probability

概率预备知识

Before embarking on the main subject, it is worth pointing out the difference of viewpoints represented by the two ensembles discussed above. In the grand canonical ensemble, the quantities of interest are defined as sums over graphs of finite size which are each attributed a positive finite weight. The scaling limit then involves adjusting the coupling constants so that large surfaces yield the dominant contribution to the statistical sums involved. This procedure enables the construction of the continuum limit of certain correlation functions, but does not construct the limiting distribution of continuum surfaces - although that would be an important achievement; see [1] and references therein. On the other hand, the infinite volume limit does involve the construction of a probability distribution of infinite CTs as a limit of distributions of finite CTs of fixed volume.

在进入正题之前，有必要先说明上文讨论的两个系综代表的不同观点。在巨正则系综中，感兴趣的量被定义为对有限尺寸图的求和，每个图都被赋予一个正的有限权重。标度极限则需要调整耦合常数，使得大曲面在相关统计和中占主导贡献。这个方法可以构造特定关联函数的连续极限，但无法构造连续曲面的极限分布——尽管这会是一项重要成果；参见文献 [1] 及其中的引用。另一方面，无限体积极限确实会将无限连续曲面 (CT) 的概率分布构造为固定体积有限 CT 分布的极限。

A simple illustration of the basic philosophy of this construction is obtained by considering standard random walk on the hypercubic integer lattice $\mathbb{Z}^d$. Here, a walk (or path) ω is a sequence (finite or infinite) of points $\omega_0, \omega_1, \omega_3, \ldots$ in $\mathbb{Z}^d$ such that ω_i and ω_{i+1} have distance 1 for all i . If ω consists of $N+1$ points, we say it has length N and denote it by $|\omega|$. One then attributes a weight $p(\omega) = (2d)^{-|\omega|}$ to a finite path ω starting at, say, the origin 0 . Considering only paths ω of a fixed length N , the weights sum up to 1, i.e., p defines a probability distribution p_N on paths of length N . Moreover, these distributions are clearly consistent for different values of N in the sense that if $N < M$ and we consider the set consisting of all paths ω' of length M coinciding with a given path ω in the first N steps, then the weights of those paths ω' add up to $p(\omega)$. This property leads in a natural way to a probability distribution on the space of infinite paths starting at 0 as follows: Given an infinite path ω , let $\mathcal{B}_{\frac{1}{N}}(\omega)$ denote the set of finite or infinite paths coinciding with ω in the first N steps and define the probability of this set to be

我们可以通过超立方整数格点 $\mathbb{Z}^d$ 上的标准随机游走，简单说明这个构造的基本思想。此处，一个游走 (或称路径) ω 是 $\mathbb{Z}^d$ 中由点 $\omega_0, \omega_1, \omega_3, \ldots$ 构成的 (有限或无限) 序列，对所有 i ，满足 ω_i 与 ω_{i+1} 的距离为 1。若 ω 包含 $N+1$ 个点，我们称其长度为 N ，记为 $|\omega|$ 。我们给从原点 0 出发的有限路径 ω 赋予权重 $p(\omega) = (2d)^{-|\omega|}$ 。如果仅考虑固定长度 N 的路径 ω ，所有权重的和为 1，即 p 在长度为 N 的路径上定义了一个概率分布 p_N 。此外，对不同的 N ，这些分布显然满足相容性: 若 $N < M$ ，我们考虑所有长度为 M 的路径 ω' ，它们在前 N 步与给定路径 ω 重合，那么这些路径 ω' 的权重之和等于 $p(\omega)$ 。这个性质自然引导我们按如下方式构造从 0 出发的无限路径空间上的概率分布: 给定一条无限路径 ω ，令 $\mathcal{B}_{\frac{1}{N}}(\omega)$ 表示所有在前 N 步与 ω 重合的有限或无限路径构成的集合，定义该集合的概率为

$$P\left(\mathcal{B}_{\frac{1}{N}}(\omega)\right) = (2d)^{-N}.$$

Recalling that p_M attributes a weight 0 to all paths not of length M , the compatibility property implies that $P\left(\mathcal{B}_{\frac{1}{N}}(\omega)\right)$ is determined by the finite size distributions as

注意到 p_M 对所有长度不是 M 的路径赋予权重 0，相容性表明 $P\left(\mathcal{B}_{\frac{1}{N}}(\omega)\right)$ 可由有限尺寸分布按如下方式确定

$$P\left(\mathcal{B}_{\frac{1}{N}}(\omega)\right) = \lim_{M\to\infty} p_M\left(\mathcal{B}_{\frac{1}{N}}(\omega)\right). \tag{1}$$

It is convenient to regard $\mathcal{B}_{\frac{1}{N}}(\omega)$ as a ball of radius $\frac{1}{N}$ around ω in the space Ω of all paths (finite or infinite) starting at 0, where the distance between any two different paths ω, ω' is defined as

我们可以方便地将 $\mathcal{B}_{\frac{1}{N}}(\omega)$ 看作所有从 0 出发的路径 (有限或无限) 构成的空间 Ω 中，围绕 ω 半径为 $\frac{1}{N}$ 的球，其中任意两条不同路径 ω, ω' 之间的距离定义为

$$d(\omega, \omega') = \frac{1}{N+1},$$

with $N = \max\{n \mid \omega_0 = \omega'_0, \ldots, \omega_n = \omega'_n\}$. It is a consequence of a rather general result on sequences of probability measures, details of which can be found in [2], that the limiting values of ball probabilities as given by (1) uniquely specify a probability distribution on Ω , thus defining an ensemble of random paths, called random walk in $\mathbb{Z}^d$. By letting N grow large in the preceding discussion, it should be clear that individual paths have vanishing P -probability. Since the set of finite paths is countable, it follows that the whole set of finite paths has vanishing P -probability, which is also expressed by saying that random walks are almost surely infinite, or that P is concentrated on infinite walks. A stronger statement is that random walks are almost surely unbounded (in $\mathbb{Z}^d$), which we leave for the reader to figure out (it follows from the fact that random walk in a finite connected graph is recurrent, in the language of section ”Definition of Spectral Dimension of Recurrent Graphs”).

根据 $N = \max\{n \mid \omega_0 = \omega'_0, \ldots, \omega_n = \omega'_n\}$ ，这是概率测度序列一个相当通用结论的推论，该结论的详细证明可参见文献 [2]: 式 (1) 给出的球概率极限值唯一确定了 Ω 上的一个概率分布，由此定义了一个随机路径的系综，称为 $\mathbb{Z}^d$ 上的随机游走。在上述讨论中令 N 趋于无穷大，不难发现单条路径的 P 概率趋近于零。由于有限路径的集合是可数的，因此全体有限路径集合的 P 概率也趋近于零，这也可以表述为随机游走几乎必然是无穷的，或者说 P 概率测度集中在无穷路径上。更强的结论是随机游走几乎必然是无界的 (在 $\mathbb{Z}^d$ 中)，这一点留给读者自行推导 (它可由“常返图谱维的定义”一节中的结论得到: 有限连通图上的随机游走是常返的)。

The strategy for constructing the infinite volume limit of generic trees in section ”Random Trees,” and of CTs in section ”Bijection Between CTs and Planar Trees,” follows the procedure sketched above quite closely. It is a recurrent theme in sections ”Hausdorff Dimension” and ”Spectral Dimension” to establish certain properties possessed by typical CTs, since the existence of atypical ones occurring with vanishing probability must be expected. A standard tool for establishing such properties is the Borel-Cantelli lemma. To formulate this result, recall first that an event in a probability space Ω is simply a subset A of Ω with an associated probability $P(A) \geq 0$. In particular, $P(\Omega) = 1$, and we say that an event A occurs almost surely (a.s.) if $P(A) = 1$, which is equivalent to $P(\Omega \setminus A) = 0$. Moreover, probabilities of mutually exclusive events add up to the probability of their union, i.e., if $A_1, A_2, \ldots$ are events such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then

在“随机树”一节构造一般树的无穷体积极限，以及在“CT 与平面树的双射”一节构造连续树 (CT) 的无穷体积极限的策略，都严格遵循了上述框架。在“豪斯多夫维数”和“谱维数”两节中，核心内容就是证明典型连续树 (CT) 具备的若干性质，因为我们必须预期非典型树出现的概率趋近于零。证明这类性质的标准工具就是博雷尔-坎泰利引理。为表述该结论，首先回顾: 概率空间 Ω 中的一个事件本质就是 Ω 的子集 A，带有对应的概率 $P(A) \geq 0$。特别地，$P(\Omega) = 1$，当 $P(A) = 1$ 时我们称事件 A 几乎必然 (a.s.) 发生，这等价于 $P(\Omega \setminus A) = 0$。此外，互斥事件的概率之和等于它们并集的概率，即如果 $A_1, A_2, \ldots$ 是满足 $A_i \cap A_j = \varnothing$ (对所有 $i \neq j$) 的事件，那么

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

In the example of random walk above, the set

在上述随机游走的例子中，集合

$$E_R = \left\{\omega : \left\|\omega_n\right\| \leq R \text{ for all } n\right\}$$

where $\|\cdot\|$ denotes the Euclidean distance to the origin in $\mathbb{Z}^d$, is the event that a walk is confined to the ball of radius R around the origin, and $\cup_{R=1}^{\infty} E_R$ is the event consisting of bounded walks.

其中 $\|\cdot\|$ 表示 $\mathbb{Z}^d$ 中到原点的欧氏距离，该集合代表游走被限制在原点周围半径为 R 的球内的事件，$\cup_{R=1}^{\infty} E_R$ 则是所有有界游走构成的事件。

Now let $A_1, A_2, \ldots$ be an arbitrary sequence of events and suppose

现在设 $A_1, A_2, \ldots$ 是任意事件序列，且假设

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

Then, since

那么，由于

$$P(A_k \cup A_{k+2} \cup A_{k+3} \ldots) \leq \sum_{n=k}^{\infty} P(A_n),$$

this implies that $P(A_k \cup A_{k+2} \cup A_{k+3} \ldots) \to 0$ as $k \to \infty$, i.e., the probability that A_n occurs for some $n \geq k$ tends to zero as k grows large. Hence, the probability that infinitely many of the events A_n happen is 0, which is the content of the Borel-Cantelli lemma.

这意味着当 $k \to \infty$ 时 $P(A_k \cup A_{k+2} \cup A_{k+3} \ldots) \to 0$，即存在某个 $n \geq k$ 使得 A_n 发生的概率随 k 增大趋于 0。因此，无穷多个事件 A_n 同时发生的概率为 0，这就是博雷尔-坎泰利引理的内容。

In our applications, we typically use the Borel-Cantelli lemma to bound the size of certain graphs. For example, suppose Ω consists of infinite graphs, and for any given n , we associate a finite graph G_n to each $G \in \Omega$ and let $|G_n|$ denote the number of edges in G_n . If (a_n) is a sequence of positive numbers and

在我们的应用中，通常使用博雷尔-坎泰利引理来界定 certain graphs 的规模。例如，假设 Ω 由无穷图组成，对任意给定的 n ，我们给每个 $G \in \Omega$ 关联一个有限图 G_n ，令 $|G_n|$ 表示 G_n 中的边数。如果 (a_n) 是一个正数列，且

$$\sum_{n=1}^{\infty} P(\{G : |G_n| > a_n\}) < \infty,$$

the Borel-Cantelli lemma implies that a.s. $|G_n| > a_n$ only for a finite number of n ' s and hence $|G_n| \leq a_n$ for n large enough with probability 1 .

博雷尔-坎泰利引理表明，$|G_n| > a_n$ 几乎必然仅对有限个 n 成立，因此当 n 足够大时，$|G_n| \leq a_n$ 以概率 1 成立。

Random Trees

随机树

Many of the discrete structures used to model quantum gravity can be viewed as graphs or graphs with some added structure. We recall that a graph G is a set of vertices $V(G)$ and a set of edges $E(G)$ which are unordered pairs of distinct vertices. In physics applications, the edges are sometimes called links. The graph is finite if the number of vertices is finite; otherwise, it is infinite. The degree (sometimes called order or valency) of a vertex v is the number of edges containing v , which will be denoted by σ_v . We say that a graph is rooted if one vertex is singled out and called the root.

许多用于量子引力建模的离散结构都可视为图，或是带有附加结构的图。我们回顾: 图 G 是由顶点集 $V(G)$ 和边集 $E(G)$ 构成的，其中边是不同顶点构成的无序对。在物理应用中，边有时也称为链接。若顶点数量有限，则图是有限的，反之则为无限图。顶点 v 的度 (有时称为阶或价) 是包含 v 的边数，记为 σ_v 。若图中指定了一个顶点作为根，则称该图是有根图。

A path of length $n \geq 1$ in a graph is a sequence of oriented edges

图中长度为 $n \geq 1$ 的路径是一组有向边构成的序列

$$(v_0, v_1), (v_1, v_2), (v_2, v_3) \ldots (v_{n-1}, v_n) \ .$$

If η is a path, we use the notation $v \in \eta$ to indicate that one of the edges in η has v as an endpoint. We say that the path is closed if $v_0 = v_n$ and simple if all the vertices v_i are different except possibly v_0 and v_n which happens when the path is closed. A simple and closed path will be called a cycle, and two cycles are considered identical if one is obtained from the other by a cyclic permutation of the edges. A trivial path consists by definition of a single vertex and is considered closed and simple and of length 0 .

若 η 是一条路径，我们使用记号 $v \in \eta$ 表示 η 中的某条边以 v 作为端点。若满足条件 $v_0 = v_n$ ，我们称该路径是闭的；若除了可能的 v_0 和 v_n 外，所有顶点 v_i 都互不相同 (路径为闭时才会出现 v_0 和 v_n 重合的情况)，我们称该路径是简单的。简单闭路径称为圈；若一个圈可由另一个圈通过边的循环置换得到，则二者被视为相同。根据定义，平凡路径仅含单个顶点，它被默认为闭且简单的，长度为 0。

A graph is connected if there is a path between any two vertices. The distance (also called graph distance) between two vertices in a connected graph is the length of the shortest path connecting them. The graph spanned by a subset V_0 of vertices of a given graph G consists of V_0 itself and those edges in $E(G)$ that connect the vertices of V_0 . The ball of radius R centered at a vertex v of G is the subgraph of G spanned by the vertices at graph distance less than or equal to R from v and will be denoted by $B_R(G; v)$. If v is the root of G , the reference to v will in general be omitted. Similarly, the boundary of $B_R(G; v)$, i.e., the subgraph spanned by the vertices at distance R from v , will be called $S_R(G; v)$. The size $|G|$ of a graph G is the number of elements in $E(G)$. The number of vertices in G will be denoted $\| G \|$, and if v is the root of G , we set $D_R(G) = \left\| S_R(G; v) \right\|$.

若任意两个顶点之间都存在路径，则图是连通的。连通图中两个顶点之间的距离 (也称为图距离) 是连接二者的最短路径的长度。给定图 G ，由顶点子集 V_0 张成的子图包含顶点集 V_0 本身，以及原图 $E(G)$ 中所有连接 V_0 顶点的边。图 G 中以顶点 v 为中心、半径 R 的球，是 G 中由所有与 v 的图距离小于等于 R 的顶点张成的子图，记为 $B_R(G; v)$ 。若 v 是 G 的根，一般会省略对 v 的标注。同理，$B_R(G; v)$ 的边界，也就是由所有与 v 距离恰好为 R 的顶点张成的子图，记为 $S_R(G; v)$ 。图 G 的大小 $|G|$ 是 $E(G)$ 中元素的数量。G 的顶点数记为 $\| G \|$ ，若 v 是 G 的根，我们记为 $D_R(G) = \left\| S_R(G; v) \right\|$ 。

We define a tree to be a connected graph which does not contain any cycle. We say that a graph is planar if it is embedded in the plane such that no two edges intersect. Note that most graphs cannot be embedded in this way, but all trees can, and a given tree can in general be embedded in many different ways. We use the term planar tree to refer to a tree with a fixed embedding up to orientation preserving homeomorphisms of the plane. Alternatively, a planar tree can be defined as a purely combinatorial object; see, for example, [3].

我们将树定义为不含任何环的连通图。若图可以嵌入平面，且任意两条边都不相交，则称该图是平面图。注意大多数图都无法这样嵌入，但所有树都可以，且一般来说一棵树有多种不同的平面嵌入方式。我们使用平面树这一术语，指代在平面保定向同胚下固定嵌入的树。另外，平面树也可以定义为纯组合对象，参见例如文献 [3]。

The Generic Random Tree

泛随机树

Let $\mathcal{T}$ be the set of all planar rooted trees, finite or infinite, such that the root, r , is of degree 1 and all vertices have finite degree. The subset of $\mathcal{T}$ consisting of trees of fixed size N will be denoted $\mathcal{T}_N$. The subset consisting of the infinite trees will be denoted $\mathcal{T}_\infty$. Given a tree $T \in \mathcal{T}$ and nonnegative integer R , the ball $B_R(T)$ is again a rooted planar tree, and its boundary $S_R(T)$ consists of $D_R(T)$ isolated vertices, whose distance R from the root will also be called their height in T . We let $T \setminus r$ denote the set of all vertices in T except the root and note that $\| T \| = |T| + 1$ for any finite tree T .

设 $\mathcal{T}$ 为所有有限或无限平面有根树的集合，其中根节点 r 的度为 1，且所有顶点的度均有限。$\mathcal{T}$ 中由固定大小 N 的树构成的子集记为 $\mathcal{T}_N$ ，由无限树构成的子集记为 $\mathcal{T}_\infty$ 。给定任意树 $T \in \mathcal{T}$ 和非负整数 R ，球 $B_R(T)$ 仍是一棵平面有根树，它的边界 $S_R(T)$ 由 $D_R(T)$ 个孤立顶点构成，这些顶点到根节点的距离 R 也称为它们在 T 中的高度。我们记 $T \smallsetminus r$ 为 T 中除根节点外所有顶点的集合，且对任意有限树 T 有 $\| T \| = |T| + 1$ 。

For later use, we define the distance $d_{\mathcal{T}}(T, T')$ between two trees T, T' as R^{-1} , where R is the radius of the largest ball around the root common to T and T' . We can view $\mathcal{T}$ as a metric space with metric $d_{\mathcal{T}}$; see [4] for some of its properties. In particular, for any positive integer R , the ball in $\mathcal{T}$ of radius R^{-1} is given by

为便于后续使用，我们定义两棵树 T, T' 之间的距离 $d_{\mathcal{T}}(T, T')$ 为 R^{-1} ，其中 R 是 T 与 T' 共有的根周围最大球的半径。我们可以将 $\mathcal{T}$ 视为带度量 $d_{\mathcal{T}}$ 的度量空间；相关性质参见文献 [4]。特别地，对任意正整数 R ，$\mathcal{T}$ 中半径为 R^{-1} 的球可表示为

$$\mathcal{B}_{\frac{1}{R}}(T) = \{T' : B_R(T') = B_R(T)\}, \tag{2}$$

i.e., it consists of all trees coinciding with T up to height R . If T has $K = D_R(T)$ vertices at height R , the trees in $\mathcal{B}_{\frac{1}{R}}(T)$ are obtained by grafting arbitrary trees

即，该球由所有到高度 R 为止都与 T 重合的树构成。如果 T 在高度 R 处有 $K = D_R(T)$ 个顶点，则 $\mathcal{B}_{\frac{1}{R}}(T)$ 中的树可通过将任意树嫁接得到

$T_1, \ldots, T_K$ onto those K vertices, i.e., identifying the root edge of T_i with the edge in T incident on the i th vertex in $S_R(T)$. In this way, there is a one-to-one correspondence between trees in $\mathcal{B}_{\frac{1}{R}}(T)$ and K -tuples of trees in $\mathcal{T}$, that we shall denote by

$T_1, \ldots, T_K$ 到这 K 个顶点上，也就是将 T_i 的根边与 T 中关联到 $S_R(T)$ 第 i 个顶点的边等同。通过这种方式，$\mathcal{B}_{\frac{1}{R}}(T)$ 中的树与 $\mathcal{T}$ 中树的 K 元组之间存在一一对应关系，我们将其记为

$$F_R : \mathcal{B}_{\frac{1}{R}}(T) \to \mathcal{T}^K \tag{3}$$

An ensemble of random graphs is a set of graphs equipped with a probability measure. In this section, we define a class of ensembles of infinite trees of relevance to quantum gravity, called generic tree ensembles, and discuss some of their properties. We proceed by first defining probability measures on finite trees of fixed size N and then showing how to obtain a limit as N tends to infinity.

随机图系综是配备了概率测度的图集合。在本节中，我们定义一类与量子引力相关的无限树系综，称为泛树系综，并讨论它们的若干性质。我们首先定义固定大小 N 的有限树上的概率测度，再说明如何在 N 趋于无穷时得到极限。

Given a set of nonnegative branching weights $w_n, n \in \mathbb{N}$, we define, in the spirit of classical statistical mechanics, the finite volume partition functions, Z_N , by

给定一组非负分支权重 $w_n, n \in \mathbb{N}$ ，我们遵循经典统计力学的思路，定义有限体积配分函数 Z_N 为

$$Z_N = \sum_{T \in \mathcal{T}_N} \prod_{v \in T \setminus r} w_{\sigma_v} \tag{4}$$

We assume $w_1 > 0$, since Z_N vanishes otherwise, and we also assume $w_n > 0$ for some $n \geq 3$ since otherwise only the linear chain of length N would contribute to Z_N . Under these assumptions the generating function ϕ for the branching weights,

> 我们假设满足 $w_1 > 0$ ，否则 Z_N 会等于零，我们还假设存在 $n \geq 3$ 使得 $w_n > 0$ 成立，否则只有长度为 N 的线性链对 Z_N 有贡献。在这些假设下，分支权重的生成函数 ϕ

$$\phi(z) = \sum_{n=1}^{\infty} w_n z^{n-1} \tag{5}$$

is strictly increasing and strictly convex on the interval $[0, z_c)$ with $\phi(0) = w_0$, where z_c is the radius of convergence for the series (5), which we assume is positive.

> 在区间 $[0, z_c)$ 上严格递增且严格凸，满足 $\phi(0) = w_0$ ，其中 z_c 是级数 (5) 的收敛半径，我们假设其为正。

It is well known (see, for example, [5]) that the generating function for the finite volume partition functions,

> 众所周知 (例如参见文献 [5])，有限体积配分函数的生成函数

$$Z(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N \tag{6}$$

satisfies the equation

> 满足方程

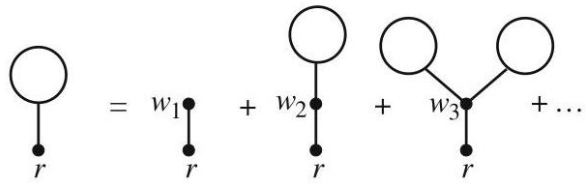
$$Z(\zeta) = \zeta \phi(Z(\zeta)), \tag{7}$$

where ζ is the fugacity associated with each edge. The proof of this identity is illustrated in Fig. 1. If the degree of the vertex next to the root is n , then there are $n-1$ trees attached to it in addition to the root edge, which has weight ζ . Summing over n then yields equation (7).

> 其中 ζ 是每条边对应的逸度。该恒等式的证明如图 1 所示。若根邻接顶点的度为 n ，则除根边外，该顶点还连接有 $n-1$ 棵树，根边的权重为 ζ 。对 n 求和即可得到式 (7)。

From the properties of ϕ , it follows that the straight line in the plane through the origin with slope ζ^{-1} intersects the graph of ϕ at least once (and at most twice) for $\zeta > 0$ small enough; see Fig. 2. By (7) and the fact that $Z(0) = 0$, it follows that

> 由 ϕ 的性质可知，当 $\zeta > 0$ 足够小时，平面上过原点、斜率为 ζ^{-1} 的直线与 ϕ 的图像至少相交一次 (至多两次)，参见图 2。结合式 (7) 与性质 $Z(0) = 0$ ，可得



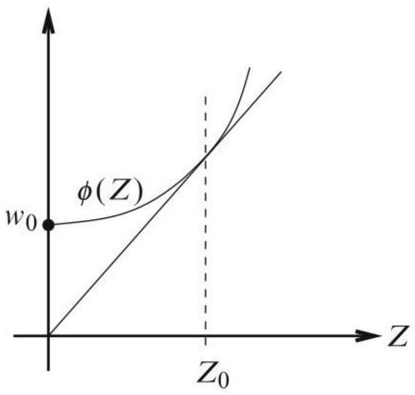


Fig. 2 An illustration of the intersection between the graph of ϕ and a straight line through the origin in the generic case

图 2 一般情形下 ϕ 的图像与过原点直线的交点示意图

Fig. 1 A graphical illustration of the derivation of equation (7)

图 1 式 (7) 推导的图示说明

$Z(\zeta)$ is determined by the first intersection point for ζ small enough and that the solution persists for $\zeta < \zeta_0$ where ζ_0 is the radius of convergence of the series (6). Since Z is an increasing function of ζ , the limit

当 ζ 足够小时，$Z(\zeta)$ 由第一个交点确定，且该解在 $\zeta < \zeta_0$ 范围内存在，其中 ζ_0 是级数 (6) 的收敛半径。由于 Z 是关于 ζ 的增函数，因此极限

$$Z_0 = \lim_{\zeta \uparrow \zeta_0} Z(\zeta) \tag{8}$$

is finite and $\leq z_c$. In the following, we consider the generic case where

有限且满足 $\leq z_c$ 。下文我们将讨论满足下述条件的一般情形:

$$Z_0 < z_c \tag{9}$$

In this case, it follows from (7) that the slope of the line through the origin that is tangent to the graph of ϕ equals ζ_0^{-1} , i.e., $Z_0 = Z(\zeta_0)$ is the unique positive solution to the equation

在该情形下，由式 (7) 可知，过原点且与 ϕ 图像相切的直线的斜率等于 ζ_0^{-1} ，即 $Z_0 = Z(\zeta_0)$ 是如下方程唯一的正解:

$$Z_0\phi'(Z_0) = \phi(Z_0). \tag{10}$$

The inequality (9) is the condition on the branching weights which singles out the generic ensembles of infinite trees to be defined below. In particular, all sets of branching weights with infinite z_c define a generic ensemble.

不等式 (9) 是分支权重需满足的条件，该条件筛选出我们下文将要定义的无限树一般系综。特别地，所有满足 z_c 为无穷大的分支权重集合都定义了一个一般系综。

In the special case $w_n = 1$ for all n , which will be encountered frequently in this article, we have by (4) that $Z_N = \sharp\mathcal{T}_N$ (we use $\sharp A$ to denote the number of elements in a set A) and evidently $\phi(z) = (1-z)^{-1}$ by (5). Solving (7) then gives

本文会经常遇到对所有 n 都满足 $w_n = 1$ 的特殊情形，由式 (4) 可得 $Z_N = \sharp\mathcal{T}_N$ (我们用 $\sharp A$ 表示集合 A 的元素个数)，且由式 (5) 显然可得 $\phi(z) = (1-z)^{-1}$ 。求解式 (7) 可得

$$Z(\zeta) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4\zeta}$$

(11)

so $\zeta_0 = \frac{1}{4}$ and $Z_0 = \frac{1}{2}$. Taylor expanding this expression now yields

因此 $\zeta_0 = \frac{1}{4}$ 且 $Z_0 = \frac{1}{2}$ 。对该表达式做泰勒展开可得

$$Z_N = C_{N-1} := \frac{(2N-2)!}{N!\,(N-1)!}, \tag{12}$$

where C_N is the N th Catalan number.

其中 C_N 是第 N 个卡特兰数。

Under the assumptions on the branching weights and assuming (9), we may, in the general case, Taylor expand ϕ around Z_0 in (7) and use (10) to conclude that the analytic function $Z(\zeta)$ has a square root branch point at ζ_0 and is given by

在分支权重满足假设且式 (9) 成立的条件下，一般情形中我们可以对式 (7) 中的 ϕ 在 Z_0 附近做泰勒展开，结合式 (10) 可得解析函数 $Z(\zeta)$ 在 ζ_0 处存在一个平方根分支点，其表达式为

$$Z(\zeta) = Z_0 - \sqrt{\frac{2\phi(Z_0)}{\zeta_0\phi''(Z_0)}}\sqrt{\zeta_0-\zeta} + O(\zeta_0-\zeta), \tag{13}$$

where the square root is chosen to be positive for $\zeta < \zeta_0$. The asymptotic behavior of Z_N is then given by

其中当 $\zeta < \zeta_0$ 时平方根取正值。Z_N 的渐近行为如下:

$$Z_N = \sqrt{\frac{\phi(Z_0)}{2\pi\phi''(Z_0)}} N^{-\frac{3}{2}} \zeta_0^{-N} \left(1 + O\left(N^{-1}\right)\right), \tag{14}$$

provided $Z_N \neq 0$. The proof of this result can be found in [6], Sections VI. 5 and VII.2. In the special case of $w_n = 1$ for all n ,(14) follows easily from (12) using Stirling's formula.

前提条件为 $Z_N \neq 0$。该结论的证明可参见文献 [6] 第 VI.5 节和 VII.2 节。当对所有 n 都满足 $w_n = 1$ 的特殊情形中，利用斯特林公式结合式 (12) 可很容易推导出式 (14)。

We define the probability distribution v_N on $\mathcal{T}_N$ by

我们在 $\mathcal{T}_N$ 上定义概率分布 v_N 如下:

$$v_N(T) = Z_N^{-1} \prod_{v \in T \setminus r} w_{\sigma_v}, \ T \in \mathcal{T}_N, \tag{15}$$

assuming $Z_N \neq 0$. Using the correspondence $F_R : \mathcal{B}_{\frac{1}{R}}(T) \to \mathcal{T}^K$ described above, where $K = D_R(T)$, the probability $v_N\left(\mathcal{B}_{\frac{1}{R}}(T)\right)$ can be expressed in terms of partition functions, and so the preceding results about the asymptotic behavior of Z_N can be applied to determine the limiting probability as $N \to \infty$. More precisely, one obtains (see Appendix A in [7] for details)

假设 $Z_N \neq 0$ 成立。利用上文描述的对应关系 $F_R : \mathcal{B}_{\frac{1}{R}}(T) \to \mathcal{T}^K$ (其中 $K = D_R(T)$)，概率 $v_N\left(\mathcal{B}_{\frac{1}{R}}(T)\right)$ 可通过配分函数表示，因此可应用此前关于 Z_N 渐近行为的结果，确定 $N \to \infty$ 时的极限概率。更确切地说，可得到 (详见文献 [7] 的附录 A)

$$\lim_{N\to\infty} v_N\left(\mathcal{B}_{\frac{1}{R}}(T)\right) = D_R(T)\, Z_0^{D_R(T)-1} \zeta_0^{|B_{R-1}(T)|} \prod_{v \in B_{R-1}(T)\setminus r} w_{\sigma_v}. \tag{16}$$

Similar to the case of random walk in $\mathbb{Z}^d$ discussed in section "Preliminary on Probability," this equals the volume of $\mathcal{B}_{\frac{1}{R}}(T)$ with respect to a limit probability measure concentrated on $\mathcal{T}_\infty$, which we denote by v . We call the ensembles $(\mathcal{T}_\infty, v)$ defined in this way generic ensembles, referring back to the genericity assumption (9). The expectation with respect to v will be denoted $\langle\cdot\rangle_v$.

类似“概率预备知识”一节中讨论的 $\mathbb{Z}^d$ 上随机游走的情形，该值等于 $\mathcal{B}_{\frac{1}{R}}(T)$ 相对于集中在 $\mathcal{T}_\infty$ 上的极限概率测度的体积，我们将该测度记为 v 。我们称由此定义的系综 $(\mathcal{T}_\infty, v)$ 为泛系综，对应于之前的一般性假设 (9)。相对于 v 的期望记为 $\langle\cdot\rangle_v$ 。

Note that if $w_n = 1$ for all n , then v_N is the uniform measure on trees of size N , i.e.,

注意若对所有 n 都满足 $w_n = 1$ ，则 v_N 是大小为 N 的树上的均匀测度，即:

$$v_N(T) = \frac{1}{C_{N-1}}, \ T \in \mathcal{T}_N. \tag{17}$$

In this case, υ is called the uniform infinite planar tree (UIPT).

在该情形下，υ 被称为均匀无限平面树 (UIPT)。

The expression (16) has a simple formal interpretation which we now explain. Given an infinite tree T , a spine for T is an infinite linear chain (non-backtracking path) in T starting at the root. We claim that υ is concentrated on the subset S of T consisting of trees with a single spine. Thus, if we denote the vertices on the spine by $s_0 = r, s_1, s_2, \dots$, ordered by increasing distance from the root, the trees in $\mathcal{S}$ are obtained by attaching branches, i.e., finite trees in $\mathcal{T}$, to each spine vertex s_n except the root, by identifying their roots with s_n . If s_n is of degree σ , there are $\sigma - 2$ branches attached to the spine at s_n . For $T \in \mathcal{S}$, we associate a weight w_{σ_v} with each vertex of T different from the root and a weight ζ_0 to each edge of T . We now argue that on a formal level these assignments characterize υ . Considering the $D_R(T) = K$ vertices of T at height $R \geq 1$, it is clear that one of them, say the i th from the left, must be s_R , and for the corresponding K -tuple $F_R(T) = (T_1, \dots, T_K)$, this means that T_i belongs to $\mathcal{S}$, while T_j is a finite tree in $\mathcal{T}$ for $j \neq i$. Using the weight assignments specified above, we obtain the total weight of single spine trees in $\mathcal{B}_R(T)$ by summing over all possible K -tuples $(T_1, \dots, T_K)$. This yields a factor Z_0 for each $j \neq i$, while the sum over T_i is to be interpreted as an integral over $\mathcal{S}$ with respect to υ which gives 1 . Moreover, summing over the position of i yields the factor $D_R(T)$, while the remaining factors in (16) arise from the part of T below height R , thus providing the desired interpretation of (16).

表达式 (16) 有一个简单的形式解释，我们在此说明。给定一棵无限树 T ，T 的主干是从根节点出发的一条无限长链 (非回溯路径)。我们指出 υ 集中在 T 的子集 S 上，该子集由仅含一条主干的树构成。因此，如果我们将主干上的顶点按距根节点距离递增的顺序记为 $s_0 = r, s_1, s_2, \dots$ ，则 $\mathcal{S}$ 中的树是通过给除根节点外的每个主干顶点 s_n 连接分支 (即 $\mathcal{T}$ 中的有限树) 得到的，连接方式是将分支的根与 s_n 重合。若 s_n 的度为 σ ，则在 s_n 处会有 $\sigma - 2$ 个分支连接到主干。对于 $T \in \mathcal{S}$ ，我们给 T 中除根节点外的每个顶点赋予权重 w_{σ_v} ，给 T 的每条边赋予权重 ζ_0 。下面我们从形式层面论证，这些赋值刻画了 υ 。考察 T 中高度为 $R \geq 1$ 的 $D_R(T) = K$ 个顶点，显然其中有一个顶点 (比如从左数第 i 个) 必定是 s_R ，对应于 K 元组 $F_R(T) = (T_1, \dots, T_K)$ ，这意味着 T_i 属于 $\mathcal{S}$ ，而对 $j \neq i$ 而言，T_j 是 $\mathcal{T}$ 中的一棵有限树。利用上述权重赋值，我们对所有可能的 K 元组 $(T_1, \dots, T_K)$ 求和，即可得到 $\mathcal{B}_R(T)$ 中仅含一条主干的树的总权重。这会给每个 $j \neq i$ 生成一个因子 Z_0 ，而对 T_i 的求和可解释为在 υ 下对 $\mathcal{S}$ 的积分，结果为 1。此外，对 i 的位置求和得到因子 $D_R(T)$ ，而 (16) 中剩余因子来自 T 高度低于 R 的部分，由此给出了 (16) 所需的解释。

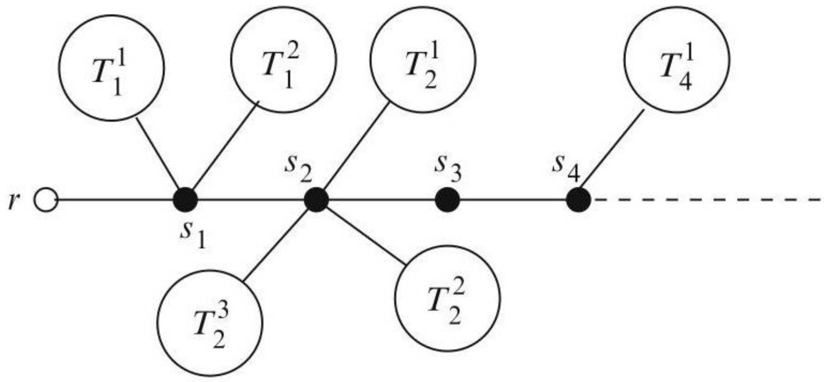
Using the above description of single spine trees in terms of (finite) branches attached at spine vertices (see Fig. 3), we can obtain the probability P that the spine vertices $s_1, \dots, s_n$ have given degrees $\sigma_1, \dots, \sigma_n$, respectively, by summing the attributed weights over the branches attached to these vertices as well as the infinite tree spanned by s_n and s_{n+1} and its descendants. Since the $\sigma_i - 2$ branches attached to s_i can be divided in $\sigma_i - 1$ ways into left and right branches and summation over individual branches yields a factor Z_0 , we obtain

利用上述单主干树 (通过在主干顶点连接有限分支得到) 的描述 (参见图 3)，我们可以得到主干顶点 $s_1, \dots, s_n$ 的度分别为给定值 $\sigma_1, \dots, \sigma_n$ 的概率 P ，方法是对连接到这些顶点的所有分支，以及由 s_n 、s_{n+1} 及其后代张成的无限树，对它们的属性权重求和。由于连接到 s_i 的 $\sigma_i - 2$ 个分支共有 $\sigma_i - 1$ 种方式划分为左分支和右分支，且对单个分支求和得到因子 Z_0 ，因此我们得到

$$P=\prod_{i=1}^{n}\zeta_0\left(\sigma_i-1\right)w_{\sigma_i}Z_0^{\sigma_i-2}.$$

Fig. 3 The first few vertices s_i on the spine of a tree and the finite branches T_i^n attached to them

图 3 一棵树主干上的前几个顶点 s_i ，以及连接到这些顶点的有限分支 T_i^n



Noting that

注意到

$$\sum_{k=2}^{\infty}\zeta_0\left(k-1\right)w_kZ_0^{k-2}=\zeta_0\phi'\left(Z_0\right)=1, \tag{18}$$

where the last equality follows from (7) and (10), this shows that the degrees of spine vertices are independently and identically distributed with probability

其中最后一个等式由式 (7) 和式 (10) 得到，这表明主干顶点的度是独立同分布的，其概率为

$$\varphi\left(k\right)=\zeta_0\left(k-1\right)w_kZ_0^{k-2} \tag{19}$$

for having degree k . Similarly, it follows from the interpretation of v just given that the individual branches T are identically and independently distributed with probability proportional to $\zeta_0^{|T|}\prod\limits_{v\in T\setminus r}w_{\sigma_v}$. The appropriate normalization factor is Z_0^{-1} yielding the probability distribution

具有度 k 。同理，由刚刚给出的 v 的解释可知，各分支 T 独立同分布，概率与 $\zeta_0^{|T|}\prod\limits_{v\in T\setminus r}w_{\sigma_v}$ 成正比。归一化因子取为 Z_0^{-1} ，由此得到概率分布

$$\mu\left(T\right)=Z_0^{-1}\zeta_0^{|T|}\prod_{v\in T\setminus r}w_{\sigma_v} \tag{20}$$

for $T\in\mathcal{T}$ finite.

当 $T\in\mathcal{T}$ 有限时。

Using (16) and (19), we can also determine the distribution of the total size of branches at a given spine vertex s_i , which will be needed in section "Spectral Dimension." Thus, denoting the union of the branches $T_i^1, \ldots, T_i^k$ at s_i by Br_i , which clearly is a tree, we define

利用式 (16) 和 (19)，我们还可以确定给定脊顶点 s_i 处分支总大小的分布，这在“谱维”一节中会用到。因此，我们将 s_i 处分支 $T_i^1, \ldots, T_i^k$ 的并集记为 Br_i ，显然它本身也是一棵树，由此定义

$$\bar{Z}_N = \nu\left(\{|Br_i| = N\}\right) = \sum_{k=0}^{\infty} \sum_{N_1+\cdots+N_k=N} \varphi(k)\, \mu\left(\left\{\left|T_i^1\right| = N_1\right\}\right) \cdots \mu\left(\left\{\left|T_i^k\right| = N_k\right\}\right),$$

where the last equality follows from the independence of the distributions of $T_i^1, \ldots, T_i^k$ and $\varphi(k)$ is given by (19). Using also (20), the corresponding generating function is given by

其中最后一个等式由 $T_i^1, \ldots, T_i^k$ 和 $\varphi(k)$ 分布的独立性得到，且由式 (19) 给出。再结合式 (20)，可得对应的生成函数为

$$\bar{Z}(s) = \sum_{N=0}^{\infty} \bar{Z}_N s^N = \sum_{k=0}^{\infty} \zeta_0 (k+1)\, w_{k+2} Z(s\zeta_0)^k = \zeta_0 \phi'\left(Z(s\zeta_0)\right).$$

As before, we may use the analyticity properties of ϕ to Taylor expand the last expression around $s = 1$ and obtain

和之前一样，我们可以利用 ϕ 的解析性，将最后一个表达式在 $s = 1$ 附近做泰勒展开，得到

$$\bar{Z}(s) = 1 - \bar{c}_0\sqrt{1-s} + O(|1-s|) \tag{21}$$

for s in a neighborhood of the unit circle, where $\bar{c}_0 > 0$ is a constant. Applying transfer theorems as before (see [6]), this implies the asymptotic behavior

对于单位圆邻域内的 s ，其中 $\bar{c}_0 > 0$ 为常数。和之前一样应用转移定理 (参见文献 [6])，可得渐近行为为

$$\bar{Z}_N = \text{ const. } N^{-3/2}\left(1 + O\left(\frac{1}{N}\right)\right).$$

for N large.

当 N 很大时。

Trees and Branching Processes

树与分支过程

We will now show how the probabilities $\mu(T)$ defined in (20) arise from a branching process. A Galton-Watson (GW) process is specified by a sequence $p_n, n = 0, 1, 2, \ldots$, of nonnegative numbers which are called offspring probabilities and satisfy

我们现在将说明 (20) 中定义的概率 $\mu(T)$ 如何从分支过程中产生。高尔顿-沃森 (GW) 过程由序列 $p_n, n = 0, 1, 2, \ldots$ 指定，该序列由称为后代概率的非负数构成，且满足

$$\sum_{n=0}^{\infty} p_n = 1 \tag{22}$$

The number p_n can be viewed as the probability of having n offspring. The process begins with one individual who has n offspring with probability p_n . Each of the offspring has n descendants with the same probability distribution, and the process continues in the same way generation after generation. Clearly, it can stop after a finite number of steps or go on forever. The motivation of Galton and Watson was to find out how likely it was that families would die out. In order to have a one-to-one correspondence between trees generated by a GW process and the tree ensembles, we have been discussing we have to assume that the first generation in the process has only one member since the root vertex has degree 1.

数 p_n 可视为产生 n 个后代的概率。该过程从一个个体开始，这个个体以概率 p_n 产生 n 个后代。每个后代都以相同的概率分布产生 n 个后代，过程逐代以相同方式延续。显然，过程可在有限步后停止，也可永远延续下去。高尔顿和沃森研究该问题的动机是探究家族灭绝的概率。为了让 GW 过程生成的树与我们一直讨论的树集合之间建立一一对应，我们必须假设过程的第一代只有一个个体，因为根顶点的度为 1。

We say that the process is critical if the mean number of offspring is 1, i.e.,

我们称当平均后代数为 1 时，过程是临界的，即

$$\sum_{n=1}^{\infty} n p_n = 1 \tag{23}$$

A critical GW process gives rise to a probability distribution π on the subset of finite trees T in $\mathcal{T}$ given by

临界 GW 过程在有限树子集 T 上得到概率分布 π ，其中 T 包含于 $\mathcal{T}$ ，由下式给出

$$\pi(T) = \prod_{i \in T \setminus r} p_{\sigma_i - 1} \tag{24}$$

as a consequence of (27) below. If we take

这是下文式 (27) 的推论。若我们取

$$p_n = \zeta_0 w_{n+1} Z_0^{n-1} \tag{25}$$

where w_n, ζ_0 , and Z_0 correspond to a generic tree as described above, then

其中 w_n, ζ_0 和 Z_0 对应上文所述的一般树，则

$$\sum_{n=0}^{\infty} p_n = \zeta_0 \sum_{n=0}^{\infty} w_{n+1} Z_0^{n-1} = \zeta_0 Z_0^{-1} \phi(Z_0) = 1 \tag{26}$$

where the last equality follows from (7). Furthermore, by (24), we have

最后一个等式由式 (7) 得到。此外，根据式 (24)，我们有

$$\pi(T) = \zeta_0^{|T|} \prod_{i \in T \setminus r} w_{\sigma_i} Z_0^{\sigma_i - 2} = Z_0^{-1} \zeta_0^{|T|} \prod_{i \in T \setminus r} w_{\sigma_i} = \mu(T), \tag{27}$$

since

由于

$$\sum_{i \in T \setminus r} (\sigma_i - 2) = -1 \tag{28}$$

for a tree T with a root of degree 1 . The reader may also easily verify that (18) is equivalent to (23), so the GW process defined by (25) is critical. Note that for the uniform tree, we have

对于根度为 1 的树 T 成立。读者也可轻易验证式 (18) 等价于式 (23)，因此式 (25) 定义的 GW 过程是临界的。注意，对于均匀树，我们有

$$p_n = 2^{-n-1} \tag{29}$$

In the following, we let f denote the generating function for the offspring probabilities given by (25):

下文中，我们令 f 表示式 (25) 给出的后代概率的生成函数:

$$f(s) = \sum_{n=0}^{\infty} p_n s^n = \zeta_0 \sum_{n=1}^{\infty} w_n Z_0^{n-2} s^{n-1} = \zeta_0 Z_0^{-1} \phi(Z_0 s). \tag{30}$$

Then equations (26) and (18) can be rewritten as

则式 (26) 和式 (18) 可改写为

$$f(1) = 1 \text{ and } f'(1) = 1. \tag{31}$$

Moreover, the genericity assumption (9) is equivalent to assuming f to be analytic in a neighborhood of the closed unit disk.

此外，一般性假设 (9) 等价于假设 f 在闭单位圆盘的邻域内解析。

If T is a finite tree, let $h(T)$ denote its height, i.e., the maximal height of vertices in T . The set of vertices at height k is called the k th generation of T , and hence $D_k(T)$ is the size of the k th generation. Clearly, $D_1 = 1$ and $P(\{D_2 = n\}) = p_n$ where $P(A)$ is the probability of the event A . Let $f_n(s)$ be the generating function for D_n , i.e.,

若 T 是一棵有限树，令 $h(T)$ 表示它的高度，即 T 中顶点的最大高度。高度为 k 的顶点集合称为 T 的第 k 代，因此 $D_k(T)$ 是第 k 代的大小。显然有 $D_1 = 1$ 和 $P(\{D_2 = n\}) = p_n$ ，其中 $P(A)$ 是事件 A 的概率。令 $f_n(s)$ 为 D_n 的生成函数，即

$$f_n(s) = \sum_{k=0}^{\infty} P(\{D_n = k\}) s^k. \tag{32}$$

Then of course $f_2(s) = f(s)$. If we assume that $D_n = k$, then the probability that $D_{n+1} = q$ is given by

那么显然有 $f_2(s) = f(s)$ 。若我们假设 $D_n = k$ ，则 $D_{n+1} = q$ 的概率由下式给出

$$P(D_{n+1} = q \mid D_n = k) = \sum_{n_1+n_2+\ldots+n_k=q} p_{n_1} p_{n_2} \cdots p_{n_k}, \tag{33}$$

so the generating function for D_{n+1} is $f_{n+1}(s) = f(f_n(s))$. By induction, it follows that f_{n+2} is the n th iterate of f .

因此 D_{n+1} 的生成函数为 $f_{n+1}(s) = f(f_n(s))$ 。由归纳法可得，f_{n+2} 是 f 的第 n 次迭代。

Clearly, the average value of D_n with respect to μ equals $f_n'(1)$. By (31) it follows that $\langle D_n \rangle_\mu = 1$ for all n . As a consequence, we get that

显然，相对于 μ ，D_n 的平均值等于 $f_n'(1)$ 。由式 (31) 可得，对所有 n 都有 $\langle D_n \rangle_\mu = 1$ 。由此我们得到

$$\langle |B_R| \rangle_\mu = \sum_{n=1}^{R} \langle D_n \rangle_\mu = R. \tag{34}$$

The probability that the GW process dies out, i.e., the tree has finite height, is given by

高尔顿-沃森过程灭绝 (即树高有限) 的概率可表示为

$$P(D_n = 0 \text{ for some } n) = \lim_{n\to\infty} P(D_n = 0) = \lim_{n\to\infty} f_n(0) \tag{35}$$

since $P(D_{n+1} = 0 \mid D_n = 0) = 1$. Since $f(0) < 1$, it is easy to see by induction that $f_n(0) < 1$ for all n . Furthermore, $f'(s) < 1$ for $0 \le s < 1$ so $f(s) > s$ for $0 \le s < 1$. It follows that $f_n(0)$ is increasing in n so the limit $\lim_{n\to\infty} f_n(0) = \lambda$ exists. Clearly, $f(\lambda) = \lambda$, and we conclude that $\lambda = 1$, so the tree has a finite height with probability 1 .

由于 $P(D_{n+1} = 0 \mid D_n = 0) = 1$ 。又因为 $f(0) < 1$ ，通过归纳法可轻松得到对所有 n 都有 $f_n(0) < 1$ 。此外，对 $0 \le s < 1$ 有 $f'(s) < 1$ ，因此对 $0 \le s < 1$ 有 $f(s) > s$ 。由此可知 $f_n(0)$ 随 n 递增，故极限 $\lim_{n\to\infty} f_n(0) = \lambda$ 存在。显然 $f(\lambda) = \lambda$ ，因此我们得到结论 $\lambda = 1$ ，即树高以概率 1 有限。

Working slightly harder, one can show that

稍加推导即可证明

$$P(D_n > 0) = \frac{2}{nf''(1)} + O\left(n^{-2}\right), \tag{36}$$

if $f''(1)$ is finite. This means that if μ is the measure on finite trees given by (20), then

当 $f''(1)$ 有限时成立。这意味着若 μ 是式 (20) 给出的有限树上的测度，则

$$\mu(\{T \in \mathcal{T} : h(T) \geq R\}) = \frac{2}{f''(1)R} + O\left(R^{-2}\right) \tag{37}$$

for R large. The proof of (36) can be found, e.g., in [8].

对大 R 成立。式 (36) 的证明可参见文献 [8]。

In the special case $p_n = bc^{n-1}$ for $n \geq 1$ and $p_0 = 1 - \sum_{n\geq 1} p_n$, the proof is simple since $b = (1-c)^2$ and

在 $p_n = bc^{n-1}$ (对应 $n \geq 1$ 和 $p_0 = 1 - \sum_{n\geq 1} p_n$) 的特殊情况下，证明非常简单，因为 $b = (1-c)^2$ 且

$$f(s) = \frac{c + (1-2c)s}{1-cs}. \tag{38}$$

The iterates of f can be calculated explicitly:

f 的迭代可以显式计算:

$$f_{n+1}(s) = \frac{(n+1)c - (nc+2c-1)s}{1+nc-(n+1)cs} \tag{39}$$

and

且

$$1 - f_{n+1}(0) = \frac{1-c}{1+nc}. \tag{40}$$

Causal Triangulations

因果三角剖分

Definition

定义

Let G be a finite rooted planar triangulation with the topology of a disk, i.e., a finite planar graph with a root S_0 such that all the faces are triangles, except one, called the exterior face, whose complement is a closed disk. We say that G is a causal triangulation (CT) if the vertices at distance k from S_0 span a cycle, i.e., the edges of $S_k(G)$ form a cycle and there are no isolated vertices in $S_k(G)$, for $0 < k < h(G)$, where

设 G 是一个具有圆盘拓扑的有限根平面三角剖分，即一个带有根 S_0 的有限平面图，其中除一个称为外面的面外，所有面均为三角形，外面的补集是一个闭圆盘。若对任意 $0 < k < h(G)$，距离根 S_0 为 k 的顶点构成一个闭合圈，即 $S_k(G)$ 的边构成一个圈且 $S_k(G)$ 中没有孤立顶点，其中

$$h(G) = \max_{v \in V(G)} d_G(S_0, v) \tag{41}$$

is called the height or radius of G. Thus, for $0 < k < h(G)$, each vertex v in $S_k(G)$ has two neighbors in $S_k(G)$ and a number $\sigma_{fv} \geq 1$ of forward neighbors in $S_{k+1}(G)$ as well as a number $\sigma_{bv} \geq 1$ of of backward neighbors in $S_{k-1}(G)$ such that

被称为 G 的高度或半径。因此，对 $0 < k < h(G)$，$S_k(G)$ 中的每个顶点 v 在 $S_k(G)$ 内有两个邻居，在 $S_{k+1}(G)$ 内有 $\sigma_{fv} \geq 1$ 个前向邻居，同时在 $S_{k-1}(G)$ 内有 $\sigma_{bv} \geq 1$ 个后向邻居，满足

$$\sigma_v = \sigma_{fv} + \sigma_{bv} + 2 \tag{42}$$

We call σ_{fv} the forward degree of v and σ_{bv} the backward degree of v, and we say v has height k in G if $v \in S_k(G)$. By convention, we shall assume that each vertex at height $h(G)$ is contained in a single triangle, thus having degree 2 and implying that boundary vertices of the disk alternate in height between $h(G)$ and $h(G) - 1$ (see Fig. 4). This boundary condition is not essential to the definition of CTs, but it is convenient when we come to defining the probability distributions on finite CTs below.

我们称 σ_{fv} 是 v 的前向度，σ_{bv} 是 v 的后向度，若满足 $v \in S_k(G)$，则称 v 在 G 中的高度为 k。按照约定，我们假设高度为 $h(G)$ 的顶点都包含在单个三角形中，因此其度数为 2，这意味着圆盘的边界顶点的高度在 $h(G)$ 和 $h(G) - 1$ 之间交替 (参见图 4)。该边界条件对因果三角剖分的定义而言不是必要的，但在我们后续定义有限因果三角剖分上的概率分布时会带来便利。

The preceding definition of finite CTs extends in a straightforward way to the case of infinite CTs, in which case $h(G) = \infty$ and no boundary is present. It is clear that any infinite CT can be drawn in such a way that it covers the whole plane, which we will generally assume in the following. Likewise, the definition above can easily be adapted to CTs with the topology of a cylinder that will be of interest in section ”Grand Canonical Ensemble and the Scaling Limit.”

上述有限因果三角剖分的定义可以直接推广到无限因果三角剖分的情形，此时 $h(G) = \infty$ 成立且不存在边界。显然，任意无限因果三角剖分都可以画成覆盖整个平面的形式，我们在下文中默认这一设定。同理，上述定义也可以轻松推广到柱面拓扑的因果三角剖分，这类三角剖分将在“巨正则系综与标度极限”一节中讨论。

We denote by C_{fin} the collection of all finite causal triangulations of the disk, by C_∞ the set of infinite triangulations of the plane, and by C their union. Moreover, let $C^{(h)}$ be the set consisting of CTs of height h. For technical reasons that will become clear below, we will always assume that one of the edges emerging from the central vertex S_0 is marked and called the root edge. In particular, this eliminates accidental symmetries under rotations around the root vertex. An example of $G \in C^4$ is shown in Fig. 4.

我们用 C_{fin} 表示所有圆盘有限因果三角剖分的集合，用 C_∞ 表示平面无限三角剖分的集合，用 C 表示二者的并集。此外，记 $C^{(h)}$ 为高度等于 h 的因果三角剖分构成的集合。出于下文将说明的技术原因，我们始终假定中心顶点 S_0 引出的一条边被标记为根边。这一约定尤其可以消除根顶点旋转带来的偶然对称性。图 4 给出了一个 $G \in C^4$ 的示例。

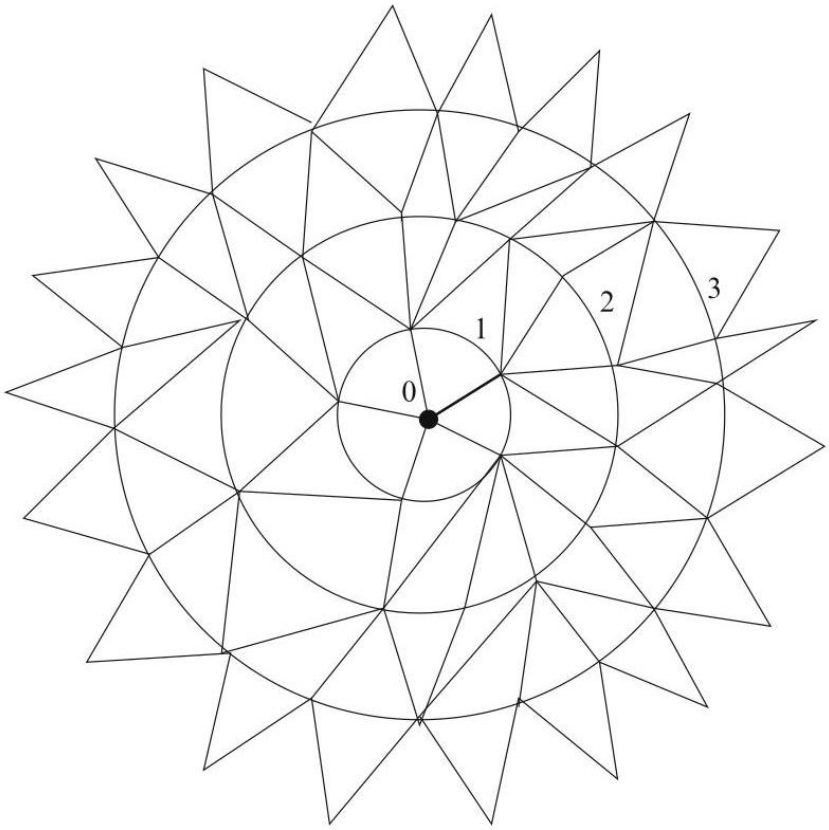


Fig. 4 Example of $G \in C^{(4)}$; the numerical labels show the heights of the cycles, and the root and marked edge are shown in bold

图 4 $G \in C^{(4)}$ 的示例；数字标号表示圈的高度，根和标记边以粗体标出

We will consider two different types of ensembles of causal triangulations. In the next section the grand canonical ensemble based on finite CTs will be defined and associated correlation functions calculated. In the present section, our main focus is on infinite CTs, making use of the results about ensembles of infinite trees in the previous section via a bijection between planar trees and CTs that we now describe.

我们将讨论两种不同类型的因果三角剖分系综。下一节会定义基于有限因果三角剖分的巨正则系综，并计算相关关联函数。本节我们将重点关注无限因果三角剖分，借助我们接下来描述的平面树与因果三角剖分之间的双射，利用上一节中关于无限树系综的结论。

Bijection Between CTs and Planar Trees

因果三角剖分 (CT) 与平面树之间的双射

Given a causal triangulation G and $k < h(G) - 1$, we will let $\sum_k(G)$ denote the subgraph of G spanned by $S_k(G)$ and $S_{k+1}(G)$, i.e., it consists of the vertices in $S_k(G)$ and $S_{k+1}(G)$ together with the edges joining them. Note that $\sum_k(G)$ is a triangulation of an annulus for $k > 0$. Denoting by $\Delta(H)$ the number of triangles in a planar graph H , we have that

给定一个因果三角剖分 G 和 $k < h(G) - 1$ ，我们用 $\sum_k(G)$ 表示由 $S_k(G)$ 和 $S_{k+1}(G)$ 张成的 G 的子图，即该子图由 $S_k(G)$ 、$S_{k+1}(G)$ 中的顶点以及连接这些顶点的边共同构成。注意当 $k > 0$ 时，$\sum_k(G)$ 是一个环带三角剖分。用 $\Delta(H)$ 表示平面图 H 中三角形的数量，我们可得:

$$\Delta\left(\sum_k\right) = |S_k| + |S_{k+1}| \tag{43}$$

and hence, due to the chosen boundary condition, we have for $G \in C_{\text{fin}}$ that

因此，根据我们选取的边界条件，对于 $G \in C_{\text{fin}}$ 有:

$$\Delta(G) = 2 \sum_{k=1}^{h(G)-1} |S_k(G)|\,. \tag{44}$$

In particular, it follows that $\Delta(G)$, which will be called the area of G , is even. We shall denote by $\mathcal{C}_N$ the subset of $\mathcal{C}_{\text{fin}}$ consisting of CTs of area $2N$ for $N \geq 1$.

特别地，可以推出 $\Delta(G)$ (我们将其称为 G 的面积) 为偶数。我们用 $\mathcal{C}_N$ 表示 $\mathcal{C}_{\text{fin}}$ 的子集，其中包含满足 $N \geq 1$ 条件、面积为 $2N$ 的因果三角剖分。

Let $G \in C_{\text{fin}}$. We define a planar rooted tree $T = \beta(G)$ from G in the following way:

设 $G \in C_{\text{fin}}$ 。我们按照如下方式从 G 构造出平面根树 $T = \beta(G)$:

1. The vertices of T are those of G whose height is at most $h(G) - 1$ together with a new vertex r which is the root of T and whose only neighbor is S_0 and which is placed in the triangle incident on the marked edge on the right as seen from S_0 .

1. T 的顶点包含 G 中高度不超过 $h(G) - 1$ 的顶点，再加上一个新顶点 r ；r 是 T 的根，它唯一的邻点是 S_0 ，放置在从 S_0 视角看、右侧标记边所在的三角形内。

2. All edges in the cycles $S_k(G), k = 1, 2, \ldots, h(G) - 1$, and those containing a vertex at maximal height are deleted, while all edges from S_0 to S_1 belong to T .

2. 删除环 $S_k(G), k = 1, 2, \ldots, h(G) - 1$ 中的所有边，以及所有包含最大高度顶点的边，而所有从 S_0 到 S_1 的边都属于 T 。

3. For each $2 \leq k < h(G) - 1$ and each vertex $v \in S_k(G)$, the rightmost of the σ_{fv} forward pointing edges as seen from v is deleted.

3. 对每个 $2 \leq k < h(G) - 1$ 和每个顶点 $v \in S_k(G)$ ，删除从 v 视角看最靠右的 σ_{fv} 条前向边。

Figure 5 shows an example of the application of these rules. Note that if the height of a vertex in G is k , then its height in $\beta(G)$ is $k+1$, i.e., the vertices in $S_{k+1}(T)$ coincide with those of $S_k(G), 0 \leq k < h(G) - 1$.

图 5 给出了应用这些规则的一个示例。注意若顶点在 G 中的高度为 k ，则该顶点在 $\beta(G)$ 中的高度为 $k+1$ ，即 $S_{k+1}(T)$ 的顶点与 $S_k(G), 0 \leq k < h(G) - 1$ 的顶点完全一致。

Conversely, let T be a rooted planar tree. Then the inverse image $G = \beta^{-1}(T)$ is obtained as follows:

反过来，设 T 为一棵根平面树，我们可以按如下方式构造它的原像 $G = \beta^{-1}(T)$:

1. Mark the rightmost edge connecting $S_1(T)$ and $S_2(T)$. Delete the root of T and the edge joining it to $S_1(T)$. The remaining vertices and edges of T all belong to G , and $S_1(T)$ becomes S_0 , the root of G .

1. 标记连接 $S_1(T)$ 和 $S_2(T)$ 的最右侧边。删除 T 的根以及根连接到 $S_1(T)$ 的边。T 剩余的顶点和边都属于 G ，而 $S_1(T)$ 成为 G 的根 S_0 。

2. For $2 \leq k \leq h(T)$, insert edges joining vertices in $S_k(T)$ in the cyclic order determined by the planarity of T ; this creates the subgraphs $S_{k-1}(G)$. (Note that by this convention we allow certain degenerate causal triangulations with cycles S_k having one or two edges corresponding to trees with one or two vertices at a given height.)

2. 对于 $2 \leq k \leq h(T)$ ，按照 T 的平面性确定的循环顺序，插入连接 $S_k(T)$ 中顶点的边；这将生成子图 $S_{k-1}(G)$ 。(注意根据该约定，我们允许某些退化因果三角剖分，其包含环 S_k ，这些环只有一条或两条边，对应给定高度下具有一个或两个顶点的树。)

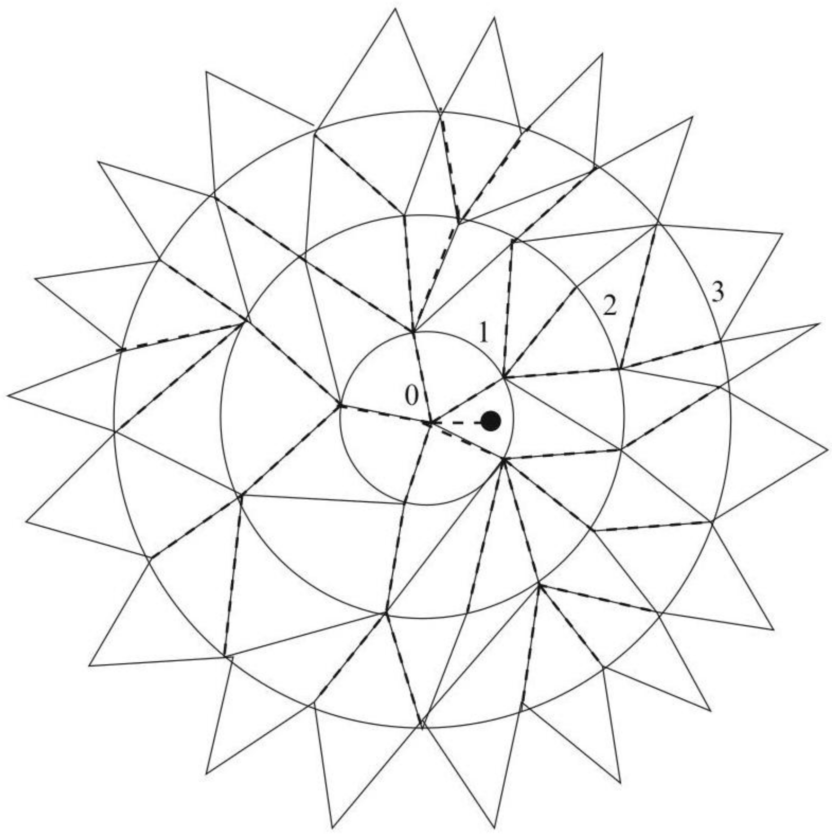


Fig. 5 The bijection from $G \in \mathcal{C}$ to $T \in \mathcal{T}$: this example shows the tree equivalent to the triangulation of Fig. 4. The dashed lines show the edges of the tree, including the new edge (r, S_0)

图 5 从 $G \in \mathcal{C}$ 到 $T \in \mathcal{T}$ 的双射: 本示例展示了与图 4 中三角剖分等价的树。虚线表示树的边，包含新边 (r, S_0)

3. For every vertex $v \in S_k(T), 2 \leq k \leq h(T) - 1$, in T draw an edge from v to a vertex in $S_{k+1}(T)$ such that the new edge is the rightmost as seen from v to $S_{k+1}(T)$ and does not cross any existing edges.

3. 对每个顶点 $v \in S_k(T), 2 \leq k \leq h(T) - 1$ ，在 T 中从 v 向 $S_{k+1}(T)$ 中的一个顶点画一条边，要求这条新边是从 v 看向 $S_{k+1}(T)$ 时最靠右的边，且不与任何已有边相交。

4. Decorate the edges of the cycle $S_{h(T)-1}(G)$ with triangles.

4. 用三角形装饰环 $S_{h(T)-1}(G)$ 的边。

A mapping equivalent to β is described in [9]. For $G \in C_{\text{fin}}$, these mappings are variants of Schaeffer's bijection [10]. Indeed, deleting the edges in $S_k(G)$ for all k and identifying the vertices of maximal height $h(G)$, one obtains a quadrangulation to which Schaeffer's bijection can be applied; here, the labeling of the vertices equals the height function. It is clear that the bijection just described extends to the case of infinite CTs and planar trees, simply by ignoring the points pertaining to the chosen boundary condition for CTs. For an extension to more general planar quadrangulations, see [3].

文献 [9] 描述了一个等价于 β 的映射。对 $G \in C_{\text{fin}}$ 而言，这些映射是谢弗双射 [10] 的变体。实际上，对所有 k 删除 $S_k(G)$ 中的边，并识别最大高度 $h(G)$ 的顶点后，可得到一个可应用谢弗双射的四边形剖分；此处顶点的标号等于高度函数。显然，我们刚刚描述的双射可以直接扩展到无限因果三角剖分和平面树的情况，只需忽略和因果三角剖分所选边界条件相关的点即可。关于向更一般平面四边形剖分的扩展，参见文献 [3]。

Using (44), this construction of β shows that it maps $\mathcal{C}_N$ bijectively onto $\mathcal{T}_{N+1}$ and likewise C_∞ onto T_∞ . Moreover, defining the metric d_C on C by

利用式 (44)，β 的这一构造表明它将 $\mathcal{C}_N$ 双射映射到 $\mathcal{T}_{N+1}$ ，同理将 C_∞ 双射映射到 T_∞ 。此外，在 C 上定义度量 d_C 为

$$d_C(G, G') = \inf\left\{\frac{1}{R+1} : B_R(G) = B_R(G')\right\}, \tag{45}$$

the map is an isometry.

该映射是一个等距映射。

Now, define the uniform finite volume probability distributions ρ_N by

现在，定义均匀有限体积概率分布 ρ_N 为

$$\rho_N(G) = \frac{1}{\sharp C_N} = \frac{1}{\sharp \mathcal{T}_{N+1}},\ G \in C_N. \tag{46}$$

Thus, we have that ρ_N is related to the uniform tree υ_N (see (17)) by

因此我们得到 ρ_N 与均匀树 υ_N (见式 (17)) 满足如下关系

$$\rho_N(G) = \upsilon_N(\beta(G)),\ G \in C_N. \tag{47}$$

It follows immediately from the existence of the UIPT υ discussed in section "The Generic Random Tree" that the limit $\rho = \lim_{N\to\infty} \rho_N$ exists and is a probability measure on C_∞ given by

由“一般随机树”一节讨论的 UIPT υ 的存在性可直接推知，极限 $\rho = \lim_{N\to\infty} \rho_N$ 存在，它是定义在 C_∞ 上的概率测度，表达式为

$$\rho(A) = \upsilon(\beta(A)) \tag{48}$$

for any event $A \subseteq \mathcal{C}_\infty$.

对任意事件 $A \subseteq \mathcal{C}_\infty$ 成立。

We call the ensemble $(\mathcal{C}_\infty, \rho)$ the uniform infinite causal triangulation (UICT). As noted in section "Random Trees," the measure υ is concentrated on the set $\mathcal{S}$ of single spine trees. Hence, ρ is concentrated on the subset $\beta^{-1}(\mathcal{S})$.

我们将总体 $(\mathcal{C}_\infty, \rho)$ 称为均匀无限因果三角剖分 (UICT)。正如“随机树”一节所述，测度 v 集中在单脊树的集合 $\mathcal{S}$ 上。因此 ρ 集中在子集 $\beta^{-1}(\mathcal{S})$ 上。

A result analogous to the above has been obtained for general planar triangulations in [11]. Finally, we observe that the relationship between trees and CTs described here is not the same as that introduced in [12] where the trees do not in general belong to a generic random tree ensemble.

文献 [11] 已经针对一般平面三角剖分得到了类似上述结论的结果。最后我们指出，本文描述的树与因果三角剖分的关系，不同于文献 [12] 中引入的关系——在文献 [12] 中，树一般不属于一般随机树总体。

Grand Canonical Ensemble and the Scaling Limit

巨正则系综与标度极限

Disk and Annulus Partition Functions

圆盘与环形配分函数

The grand canonical ensemble for finite CTs was introduced in [13]. The disk partition function W_M for CTs of a fixed height h is defined by assigning each triangle in $G \in \mathcal{C}^{(h)}$ (see Fig. 4) a weight g and each boundary triangle an additional weight factor yg^{-1} ; this gives

有限因果三角剖分的巨正则系综最早见于文献 [13]。对固定高度 h 的因果三角剖分，定义圆盘配分函数 W_M 的方法是: 给 $G \in \mathcal{C}^{(h)}$ (见图 4) 中每个三角形赋予权重 g ，给每个边界三角形额外赋予权重因子 yg^{-1} ，由此得到

$$W_M(g,z;h) = \sum_{G\in C^{(h)}} |S_{h-1}(G)|\,(z/g)^{|S_{h-1}(G)|} g^{\Delta(G)}$$
$$= z\frac{\partial}{\partial z} \sum_{G\in C^{(h)}} (z/g)^{|S_{h-1}(G)|} g^{\Delta(G)}. \tag{49}$$

Here, the subscript M indicates that the disk boundary is marked (recall that there is also a marked root edge), which generates the factor $|S_{h-1}(G)|$ in the weight of $G.W_M$ can be thought of as the discretized path integral for the amplitude that a Euclidean universe with disk topology starts at a point S_0 at Euclidean time 0 and has a single connected boundary at Euclidean time h . Then $\log g$ is the bulk cosmological constant coupled to $\Delta(G)$ which is the space time volume, and $\log y$ is the boundary cosmological constant coupled to the boundary length given by the number of boundary triangles, $|S_{h-1}(G)|$.

此处下标 M 表示圆盘边界被标记 (注意还存在一条标记的根边)，由此产生因子 $|S_{h-1}(G)|$ ；权重中的 $G.W_M$ 可以理解为离散路径积分的振幅: 具有圆盘拓扑的欧几里得宇宙在欧几里得时间 0 始于点 S_0 ，在欧几里得时间 h 处有一条单连通边界。其中 $\log g$ 是与时空体积 $\Delta(G)$ 耦合的体宇宙学常数，$\log y$ 是与边界长度 (由边界三角形的数量给出，即 $|S_{h-1}(G)|$) 耦合的边界宇宙学常数。

Correspondingly, the annulus (or cylinder) amplitude describes a Euclidean universe that evolves in Euclidean time h from an entrance boundary to an exit boundary. The contributing graphs are created from the disk graphs by inserting a second boundary at height 0 ; starting with $G \in C^{(h)}$, separate the triangles in $\sum_0$ so that they no longer have edges in common but still have an edge in S_1 . The resulting entrance boundary contains $|S_1|$ triangles. Each is assigned an extra weight factor xg^{-1} , and one, defined to be the triangle immediately clockwise of the marked edge in G , is marked. The annulus partition function with one marked triangle on the exit boundary is then

相应地，环形 (或柱形) 振幅描述欧几里得宇宙在欧几里得时间 h 内从入口边界演化到出口边界的过程。这类图可由圆盘图构造得到: 在高度 0 处插入第二条边界，从 $G \in C^{(h)}$ 出发，将 $\sum_0$ 中的三角形分离，使它们不再共享边，但仍在 S_1 中保留一条边。最终得到的入口边界包含 $|S_1|$ 个三角形，每个三角形被赋予额外权重因子 xg^{-1} ， 其中有一个三角形 (定义为 G 中标记边顺时针方向的下一个三角形) 被标记。由此，出口边界上带有一个标记三角形的环形配分函数为

$$W_{MM}(g,x,y;h) = \sum_{G\in C^{(h)}} (x/g)^{|S_1(G)|}\,|S_{h-1}(G)|\,(y/g)^{|S_{h-1}(G)|} g^{\Delta(G)}. \tag{50}$$

The partition functions are computed using the bijective map $\beta : C^{(h)} \to \mathcal{T}^{(h)}$ rather easily. Let $w_h(g,z)$ be the partition function for trees of height $\le h$, with each vertex v assigned a weight $g^{2(\sigma_v-1)}$, and each vertex at height h assigned a further weight zg^{-1} , then

利用双射 $\beta : C^{(h)} \to \mathcal{T}^{(h)}$ 可以很容易地计算配分函数。设高度为 $\le h$ 的树的配分函数为 $w_h(g,z)$ ，每个顶点 v 赋予权重 $g^{2(\sigma_v-1)}$ ， 每个高度为 h 的顶点额外赋予权重 zg^{-1} ， 则有

$$w_h(g,z) = \sum_{h'\le h}\sum_{T\in\mathcal{T}^{(h')}} (z/g)^{\|S_h(T)\|}\left(\prod_{v\in T\setminus r} g^{2(\sigma_v-1)}\right). \tag{51}$$

Each vertex in $S_{i+1}(T)$ has exactly one edge connecting it to a vertex in $S_i(T)$ for $i = 1 \dots h(T)-1$. So every vertex $v \in T\setminus r$ contributes $\sigma_v - 1$ vertices in $G \setminus S_{h(G)}(G)$, where $G = \beta^{-1}(T)$, and thus, using (44),

$S_{i+1}(T)$ 中每个顶点恰好有一条边连接到 $i = 1 \dots h(T)-1$ 对应 $S_i(T)$ 中的一个顶点。因此每个顶点 $v \in T\setminus r$ 都会在 $G \setminus S_{h(G)}(G)$ 中贡献 $\sigma_v - 1$ 个顶点 (满足 $G = \beta^{-1}(T)$)，由此结合式 (44) 可得

$$\Delta(G) = 2\sum_{v\in T\setminus r}(\sigma_v - 1). \tag{52}$$

Using the map β to rewrite the right-hand side of (51) as a sum over CTs gives

利用映射 β 将式 (51) 的右侧改写为对因果三角剖分的求和，得到

$$w_h(g,z) = \sum_{h'\le h}\sum_{G\in\mathcal{C}^{(h')}} (z/g)^{|S_{h-1}(G)|} g^{\Delta(G)}. \tag{53}$$

Only trees T of height h generate z -dependent contributions to (51), so differentiating the right-hand side of (51) w.r.t. z suppresses all except the $h' = h$ terms; hence

只有高度为 h 的树 T 会对式 (51) 产生与 z 相关的贡献，因此对式 (51) 的右侧关于 z 求导可以消去除 $h' = h$ 项以外的所有项，故有

$$z\frac{\partial}{\partial z}w_h(g,z) = W_M(g,z;h). \tag{54}$$

To compute $w_h(g,z)$, we decompose the trees of height h into trees of height $h-1$ by cutting at the vertex adjacent to the root (see Fig. 6), which gives

为计算 $w_h(g,z)$ ，我们在紧邻根的顶点处切割 (见图 6)，将高度为 h 的树分解为多个高度为 $h-1$ 的树，由此得到

$$w_h(g,z) = \sum_{k=0}^{\infty} g^{2k}(w_{h-1}(g,z))^k = \frac{1}{1-g^2 w_{h-1}(g,z)}, \tag{55}$$

with $w_1(g,z) = zg^{-1}$. This recursion is easily solved by setting $w_h = u_h/u_{h+1}$ which gives a linear difference equation for u_h ; imposing the initial condition, and choosing the convenient parametrization $g^{-1} = 2\cosh\theta$, leads to

在 $w_1(g,z) = zg^{-1}$ 条件下。通过设定 $w_h = u_h/u_{h+1}$ 可以轻松求解该递推关系，该设定会得到一个关于 u_h 的线性差分方程；施加初始条件并选取方便的参数化形式 $g^{-1} = 2\cosh\theta$ 后，可得

$$w_h(g,z) = 2\cosh\theta\frac{\sinh(h-1)\theta - z\sinh(h-2)\theta}{\sinh h\theta - z\sinh(h-1)\theta}, \tag{56}$$

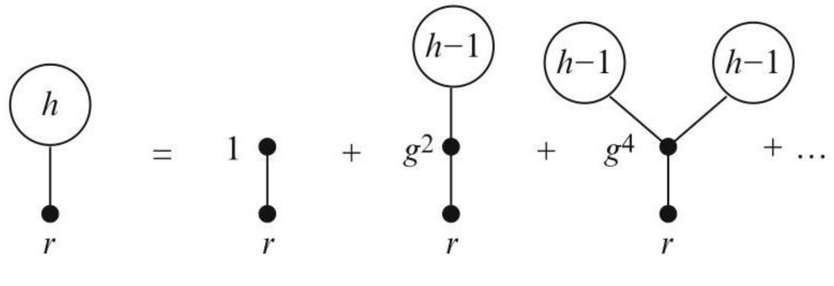
and hence the disk partition function is

因此圆盘配分函数为

$$W_M(g,z;h) = 2\cosh\theta\frac{z\sinh^2\theta}{(\sinh h\theta - z\sinh(h-1)\theta)^2}. \tag{57}$$

Fig. 6 Graphical representation of equation (55)

图 6 方程 (55) 的图示表示



To find the annulus partition function, we follow the same steps until the final iteration of the tree recurrence. Here, each offspring of the vertex adjacent to the root has a weight $g^2(x/g) = xg$ instead of g^2 , so

为了得到环带配分函数，我们遵循相同步骤处理树递推的最后一次迭代。此处，与根相邻顶点的每个后代的权重为 $g^2(x/g) = xg$ 而非 g^2 ，因此

$$W_{MM}(g, x, y; h) = y\frac{\partial}{\partial y}\frac{1}{1 - xgw_{h-1}(g, y)}$$

$$= \frac{xy\sinh^2\theta}{\left(\sinh(h-1)\theta - (x+y)\sinh(h-2)\theta + xy\sinh(h-3)\theta\right)^2}.$$

(58)

The partition functions (57) and (58) for every h are analytic functions of g, x , and y in the region

对任意 h ，配分函数 (57) 和 (58) 是区域内 g, x 和 y 的解析函数

$$A : |g| < g_c = \frac{1}{2}, |x| < 1, |y| < 1. \tag{59}$$

Note that for a given finite h and $g < \frac{1}{2}$ the poles of W_{MM} in x and y lie strictly outside A .

注意，对于给定的有限 h 和 $g < \frac{1}{2}$ ，W_{MM} 在 x 和 y 中的极点严格位于 A 外部。

Finally, we remark that $\frac{1}{2}w_n\left(g = \frac{1}{2}, s\right)$ is the offspring probability generating function $f_{n-1}(s)$ for the uniform random tree, given by (39) with $c = \frac{1}{2}$.

最后我们注意，$\frac{1}{2}w_n\left(g = \frac{1}{2}, s\right)$ 是均匀随机树的后代概率生成函数 $f_{n-1}(s)$ ，由 (39) 式在 $c = \frac{1}{2}$ 条件下给出。

Scaling Amplitudes

标度振幅

As noted above, the partition functions (57) and (58) are analytic in the region A . Within A , the partition functions are dominated by graphs with small area and short boundaries. Approaching the limits of A , the area and boundary length(s) of the dominant graphs grow arbitrarily large, and the scaling limit can be constructed. Expanding (57) about $\theta = 0$ at fixed h and $y < 1$ gives

如上所述，配分函数 (57) 和 (58) 在区域 A 内解析。在 A 内部，配分函数由小面积、短边界的图主导。当趋近 A 的边界时，主导图的面积和边界长度会任意增大，由此可以构造标度极限。在固定 h 和 $y < 1$ 时，对 (57) 在 $\theta = 0$ 附近展开可得

$$W_M(g, y, h) = 2\frac{y}{\left(h(1-y) + y\right)^2} + O\left(\theta^2\right), \tag{60}$$

which reflects the fact that tall trees are rare even at $g = \frac{1}{2}$. The universe described by W_M does not survive when $h \to \infty$ unless the limit is taken in such a way that $h(1-y) = \text{const}$; only then does the model generate universes which are very large, compared to the discretization scale, in the Euclidean time direction. The physically nontrivial limit is obtained by setting $g = \frac{1}{2}\operatorname{sech}\theta, y = 1 - Y\theta\Lambda^{-\frac{1}{2}}, h = H\theta^{-1}\Lambda^{\frac{1}{2}}$ and taking $\theta \to 0$. (A mathematical treatment of the weak convergence properties of this limit is given in [14].) The scaling amplitudes are then defined to be

这反映了即使在 $g = \frac{1}{2}$ 处高树也十分稀少的事实。除非按 $h(1-y) =$ 为常数的方式取极限，否则由 W_M 描述的宇宙无法在 $h \to \infty$ 条件下存续；只有这样，模型才能生成在欧几里得时间方向上远大于离散化尺度的大宇宙。通过设定 $g = \frac{1}{2}\operatorname{sech}\theta, y = 1 - Y\theta\Lambda^{-\frac{1}{2}}, h = H\theta^{-1}\Lambda^{\frac{1}{2}}$ 并取 $\theta \to 0$ ，即可得到非平凡的物理极限。(该极限弱收敛性质的数学处理参见文献 [14]。) 由此定义标度振幅为

$$W_M^s(\Lambda, Y, H) = \lim_{\theta\to 0} W_M\left(\frac{1}{2}\operatorname{sech}\theta, 1 - Y\theta\Lambda^{-\frac{1}{2}}, H\theta^{-1}\Lambda^{\frac{1}{2}}\right)$$

$$= 2\frac{\Lambda}{\left(\Lambda^{\frac{1}{2}}\cosh H\Lambda^{\frac{1}{2}} + Y\sinh H\Lambda^{\frac{1}{2}}\right)^2}, \tag{61}$$

and

且

$$W_{MM}^s(\Lambda, X, Y, H) = \lim_{\theta\to 0}\theta^2 W_{MM}\left(\frac{1}{2}\operatorname{sech}\theta, 1 - X\theta\Lambda^{-\frac{1}{2}}, 1 - Y\theta\Lambda^{-\frac{1}{2}}, H\theta^{-1}\Lambda^{\frac{1}{2}}\right)$$

$$= \frac{\Lambda}{\left((\Lambda + XY)\sinh\Lambda^{\frac{1}{2}}H + \Lambda^{\frac{1}{2}}(X+Y)\cosh\Lambda^{\frac{1}{2}}H\right)^2}.$$

(62)

The pre-factor θ^2 in the definition of W_{MM}^s reflects the insertion of an extra marked boundary relative to W_M^s which renders the partition function divergent at the critical point. W_M^s is the amplitude for a continuum Euclidean universe with disk topology starting at Euclidean time 0 and having a boundary at Euclidean time H with bulk cosmological constant Λ and boundary cosmological constant Y ; similarly, W_{MM}^s describes a universe with, in addition, a boundary at time 0 having boundary cosmological constant X.H is chosen to have length dimension $[H] = 1$, so $[\Lambda] = -2$, and the extents of the boundaries conjugate to X and Y also have dimension 1. The scaling dimension d_H (sometimes called the scaling Hausdorff dimension) is defined through the dependence of the average area of graphs $\langle\Delta(G)\rangle_\theta$ on the height $h(\theta) = H\theta^{-1}\Lambda^{\frac{1}{2}}$ in the scaling limit of the disk ensemble

W_{MM}^s 定义中的前置因子 θ^2 反映了，相较于 W_M^s 额外插入了一个标记边界，这使得配分函数在临界点发散。W_M^s 是连续欧几里得宇宙的振幅，该宇宙为圆盘拓扑，从欧几里得时间 0 开始，在欧几里得时间 H 处存在边界，具有体宇宙学常数 Λ 和边界宇宙学常数 Y ；类似地，W_{MM}^s 描述的宇宙额外在时间 0 处存在一个边界，其边界宇宙学常数为 X.H 。我们选取 X.H 具有长度量纲 $[H] = 1$ ，因此 $[\Lambda] = -2$ ，而与 X 和 Y 共轭的边界范围也具有量纲 1。标度维数 d_H (有时称为标度豪斯多夫维数) 由圆盘系综标度极限中，平均图面积 $\langle\Delta(G)\rangle_\theta$ 对高度 $h(\theta) = H\theta^{-1}\Lambda^{\frac{1}{2}}$ 的依赖关系定义

$$d_H = \lim_{\theta\to 0}\frac{\log\langle\Delta(G)\rangle_\theta}{\log h(\theta)}, \tag{63}$$

where

其中

$$\langle \Delta(G) \rangle_\theta = g \frac{\partial}{\partial g} \log W_M(g, y(\theta); h(\theta)) \Big|_{g=\frac{1}{2}\operatorname{sech}\theta, y(\theta)=1-Y\theta\Lambda^{-\frac{1}{2}}}. \tag{64}$$

This gives $d_H = 2$ which is consistent with the dimension of the spatial and temporal extents each being 1 and the universes described by the scaling limit being colloquially two-dimensional. See [15] for further discussion of these partition functions.

由此得到 $d_H = 2$，这与空间范围和时间范围各为 1 维、标度极限描述的宇宙通俗来讲是二维的结论一致。关于这些配分函数的进一步讨论参见文献 [15]。

Hausdorff Dimension

豪斯多夫维数

In the previous section, we introduced the dimension d_H which relates the total area and the linear extent in the limit when both become large. This section takes another point of view; we consider infinite graphs and the relation between the size of a ball and its radius as the latter becomes large.

在上一节中，我们介绍了维数 d_H，它描述了二者趋于无穷大的极限下总面积与线性尺度的关系。本节将采用另一视角: 我们考虑无穷图，以及半径趋于无穷大时球的大小与半径的关系。

The Hausdorff dimension d_h (sometimes called the local Hausdorff dimension) of an infinite rooted graph G is defined by the relation

无穷根图 G 的豪斯多夫维数 d_h (有时也称为局部豪斯多夫维数) 由下式定义

$$|B_R(G)| \sim R^{d_h}, \ R \to \infty, \tag{65}$$

where B_R as usual denotes the ball of radius R around the root and $|B_R|$ its size. More precisely, we define

其中 B_R 按惯例表示根周围半径为 R 的球，$|B_R|$ 是该球的大小。更严谨地，我们定义

$$d_h = \lim_{R\to\infty} \frac{\ln |B_R(G)|}{\ln R} \tag{66}$$

whenever the limit exists. For the ensembles of trees and surfaces that are studied here, we show that this is indeed the case and yields the same value of d_h for almost all G . We shall likewise see that the same value of d_h is obtained by replacing $|B_R(G)|$ in (66) by its average value, which is in general easier to evaluate or estimate.

极限存在时该定义有效。对于本文研究的树与曲面系综，我们证明极限确实存在，且几乎对所有 G 都给出相同的 d_h 值。我们还会看到，将 (66) 式中的 $|B_R(G)|$ 替换为其平均值也能得到相同的 d_h 值，而平均值通常更便于计算和估计。

In many important cases, $d_h = d_H$; this includes the ensembles studied in this paper, but the relation does not hold universally as will be seen in section ”Curvature and Matter Fields on the CDT,” albeit in a case where the ensemble weights may take negative values.

在许多重要情形中都有 $d_h = d_H$ ；这包含了本文研究的系综，但该关系并非普遍成立，我们会在“CDT 上的曲率与物质场”一节看到反例，只不过该反例中的系综权重可能取负值。

Generic Trees

泛树

Let T be a generic tree with associated probability distribution υ . We can assume, as has been explained in section ”Random Trees,” that T has a unique spine. Let $T_i^n, n = 0, 1, \ldots \sigma_{s_i} - 2$ be the finite trees attached to the i th vertex on the spine (see Fig. 3) and recall that these are independent and each is distributed according to the probability measure μ given by (20). With notation as in section ”The Generic Random Tree,” we have

设 T 为一棵带有关联概率分布 υ 的泛树。正如“随机树”一节中所述，我们可以假设 T 存在唯一主干。设 $T_i^n, n = 0, 1, \ldots \sigma_{s_i} - 2$ 是附在主干上第 i 个顶点的有限树 (参见图 3)，请注意这些树相互独立，且每个都服从式 (20) 给出的概率测度 μ 。采用“泛随机树”一节中的记号，我们有

$$Br_i = \bigcup_{n=1}^{\sigma_{s_i}-2} T_i^n \tag{67}$$

interpreted as the empty graph if $\sigma_{s_i} = 2$. Letting Y_j denote the number of vertices different from s_i in Br_i located at distance $\leq j$ from s_i , we can write

当 $\sigma_{s_i} = 2$ 时，该式被解释为空图。令 Y_j 表示 Br_i 中与 s_i 不同、且距离 s_i 为 $\leq j$ 的顶点数，我们可以写出

$$|B_R| = R + \sum_{i=1}^{R} Y_{R-i} \tag{68}$$

where the R on the right-hand side accounts for the number edges on the spine inside B_R . It follows from (19) and (25) that

其中右侧的 R 对应 B_R 内主干上的边数。由式 (19) 和 (25) 可得

$$\upsilon\left(\{\sigma_{s_i} = n+2\}\right) = (n+1)\, p_{n+1}. \tag{69}$$

When multiplied by n , this is the $(n-1)$ th Taylor coefficient of f'' . Using (69) and (34), this gives

该式乘 n 后，就是 f'' 的第 $(n-1)$ 项泰勒系数。结合式 (69) 和 (34)，可得

$$\langle Y_{R-i}\rangle_{\upsilon} = f''(1)(R-i). \tag{70}$$

Summing over i from 1 to R yields

对 i 从 1 到 R 求和得到

$$\langle |B_R|\rangle_{\upsilon} = \frac{1}{2}f''(1)R(R-1)+R, \tag{71}$$

which shows that in terms of average values of ball sizes we have $d_h = 2$.

这表明，就球尺寸的平均值而言，我们有 $d_h = 2$ 。

To obtain bounds on $|B_R(T)|$ for individual trees is more cumbersome, and we shall not elaborate in detail on this issue here. In section "Spectral Dimension of Generic Trees," we show that if $G_R = \cup_{i=1}^{R} Br_i$, then

对单棵树的 $|B_R(T)|$ 取界要更复杂，本文在此不展开详细讨论。在“泛树的谱维数”一节中我们证明，若 $G_R = \cup_{i=1}^{R} Br_i$ ，则

$$|G_R| \le C_2 R^2 (\ln R)^3$$

holds for R large enough almost surely with respect to υ (see (110)). Since we clearly have $B_R \subseteq G_R$, the same bound holds for $|B_R|$. A similar lower bound is shown in [16] yielding the a.s. bounds

当 R 足够大时，该式关于 υ 几乎必然成立 (参见式 (110))。由于显然有 $B_R \subseteq G_R$ ，该界对 $|B_R|$ 同样成立。文献 [16] 给出了一个类似的下界，由此得到几乎必然界

$$C_1 (\ln R)^{-2} R^2 \le |B_R(T)| \le C_2 R^2 (\ln R)^3, \tag{72}$$

where C_1 and C_2 are positive constants. Evidently, these bounds imply that $d_h = 2$ a.s.

其中 C_1 和 C_2 为正的常数。显然，这些界意味着 $d_h = 2$ 几乎必然成立。

It is worth remarking that the ensemble average of the volume of a ball $B_R(\upsilon)$ centered at a random vertex υ within some fixed distance from the root displays the same behavior as in (72) as a simple consequence of the triangle inequality.

值得一提的是，根据三角不等式，根的固定距离范围内随机顶点 υ 处中心在该点的球 $B_R(\upsilon)$ ，其体积的系综平均表现出与式 (72) 相同的行为。

Causal Triangulations

因果三角剖分

We now turn to the Hausdorff dimension of causal triangulations. For an infinite causal triangulation G , we have

我们现在转而研究因果三角剖分的豪斯多夫维数。对于无穷因果三角剖分 G ，我们有

$$|B_R(G)| = 2\sum_{i=1}^{R} |S_i| + \sum_{i=1}^{R-1} |S_i| \tag{73}$$

and it follows that

由此可得

$$\left\|B_R\right\| \leq |B_R| \leq 3\left\|B_R\right\|. \tag{74}$$

Clearly, $\left\|B_R\right\|+1$ equals the number of vertices within distance R from the root of the tree T corresponding to G under the bijection β . Hence, in view of (71),

显然，在双射 β 下，$\left\|B_R\right\|+1$ 等于对应于 G 的树 T 的根周围距离 R 范围内的顶点数。因此，结合式 (71),

$$\langle |B_R| \rangle_\rho \sim R^2,\ R \to \infty, \tag{75}$$

where the expectation is with respect to the measure ρ defined in (48), so $d_h = 2$ for CTs. By the same argument, we likewise have a.s. that

其中期望是对式 (48) 定义的测度 ρ 而言的，因此对因果三角剖分有 $d_h = 2$ 。通过相同推导，我们同样几乎必然得到

$$C_1'(\ln R)^{-2} R^2 \leq |B_R(G)| \leq C_2' R^2 (\ln R)^3, \tag{76}$$

and hence that $d_h = 2$ a.s. with respect to ρ .

因此关于测度 ρ 几乎必然有 $d_h = 2$ 。

Spectral Dimension

谱维数

In this section, we define a notion of dimension for graphs which is different from the ones discussed above. This is the spectral dimension which is a measure of how likely it is that a random walker returns to the starting point. In the following subsections, we analyze the relation between the spectral and Hausdorff dimensions and calculate the spectral dimension for generic trees and causal triangulations.

在本节中，我们定义一个不同于上文讨论的图维度概念，即谱维数，它用于衡量随机游走者回到起点的概率。在接下来的小节中，我们分析谱维数与豪斯多夫维数之间的关系，并计算通用树和因果三角剖分的谱维数。

Definition of Spectral Dimension of Recurrent Graphs

常返图的谱维数定义

Given a graph G , we use the notation $\omega : v \to u$ to indicate a path ω from vertex v to vertex u , and if ω has length $|\omega| = m$, the vertices of ω will be denoted by $v = \omega_0, \omega_1, \ldots, \omega_{m-1}, \omega_m = u$. If $v \neq u$, we write $\omega : v \to u$ for a path from v to u that does not return to v , i.e., $\omega_i \neq v$ for $i \neq 0$. If $v = u$, the notation $\omega : v \to v$ is used for a path from v to v that does not return to v in between, i.e., $\omega_i \neq v$ for $i = 1, 2, \ldots, |\omega| - 1$. Below, we also consider infinite paths $\omega = (\omega_0\omega_1), (\omega_1\omega_2), (\omega_2\omega_3), \ldots$ emerging from a vertex $v = \omega_0$.

给定图 G ，我们使用记号 $\omega : v \to u$ 表示一条从顶点 v 到顶点 u 的路径 ω ，若 ω 的长度为 $|\omega| = m$ ，则 ω 的顶点记为 $v = \omega_0, \omega_1, \ldots, \omega_{m-1}, \omega_m = u$ 。当 $v \neq u$ 时，我们用 $\omega : v \to u$ 表示一条从 v 到 u 且不返回 v 的路径，即对 $i \neq 0$ 满足 $\omega_i \neq v$ 。当 $v = u$ 时，记号 $\omega : v \to v$ 表示一条从 v 出发回到 v 且中途不返回 v 的路径，即对 $i = 1, 2, \ldots, |\omega| - 1$ 满足 $\omega_i \neq v$ 。在下文中，我们还会讨论从顶点 $v = \omega_0$ 出发的无穷路径 $\omega = (\omega_0\omega_1), (\omega_1\omega_2), (\omega_2\omega_3), \ldots$ 。

We define the function p_G on the set of all finite paths on G by

我们在 G 上所有有限路径的集合上定义函数 p_G 为

$$p_G(\omega) = \prod_{i=0}^{|\omega|-1} \sigma_{\omega_i}^{-1}$$

It is easily seen that p_G defines a probability distribution on the set $\Pi_m(v)$ of paths of fixed length m and fixed initial vertex v and that these distributions are compatible in the way described in section "Preliminary on Probability." We define a probability distribution $P_{G,v}$ on the set $\Pi_\infty(v)$ of all infinite paths starting at v by setting

不难验证，p_G 在固定长度 m 、固定起点顶点 v 的路径集合 $\Pi_m(v)$ 上定义了一个概率分布，且这些分布满足“概率预备知识”一节中所述的相容性。我们通过以下设定，在所有从 v 出发的无穷路径集合 $\Pi_\infty(v)$ 上定义概率分布 $P_{G,v}$

$$P_{G,v}(A(\overline{\omega})) = p_G(\overline{\omega}),$$

where $\overline{\omega}$ is an arbitrary finite path starting at v and $A(\overline{\omega})$ denotes the set of all infinite paths that coincide with $\overline{\omega}$ in the first $|\overline{\omega}|$ edges. When considered as probability spaces in the way described, the paths in $\Pi_m(v)$ or $\Pi_\infty(v)$ are usually referred to as random walks. (More commonly, they are called simple random walks, to distinguish them from, e.g., biased random walks. Since we do not consider different kinds of random walks in this paper we will leave out the adjective ”simple”.)

其中 ϖ 是任意一条从 v 出发的有限路径，$A(\varpi)$ 表示前 $|\varpi|$ 条边与 ϖ 重合的所有无穷路径构成的集合。按上述方式定义概率空间后，$\Pi_m(v)$ 或 $\Pi_\infty(v)$ 中的路径通常被称为随机游走。(更常见的叫法是简单随机游走，用以区别带偏随机游走等其他类型。由于本文不讨论其他类型的随机游走，我们省略“简单”二字。)

The probability for a random walk of length m starting at v to end at u is given by

起点为 v 、长度为 m 的随机游走最终到达 u 的概率为

$$q_G(m; v, u) = \sum_{\omega : v \to u, |\omega| = m} p_G(\omega).$$

The corresponding cumulative probability is defined as

对应的累积概率定义为

$$Q_G(n; v, u) = \sum_{m=0}^{n} q_G(m; v, u)$$

for any $n = 0, 1, 2, \dots$, with the convention

对任意 $n = 0, 1, 2, \dots$ 成立，约定

$$q_G(0; v, u) = \delta_{v,u} = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{if } v \neq u, \end{cases}$$

i.e., we define $p_G(\omega) = 1$ for the trivial walk ω of length 0 consisting of a single vertex.

即我们对仅含单个顶点、长度为 0 的平凡游走 ω 定义了 $p_G(\omega) = 1$ 。

With this convention, we note for later reference that Q_G fulfills

根据该约定，我们为后文参考记为 Q_G 满足

$$Q_G(n+1; v, u) = \sum_{x : (u,x) \in E(G)} \sigma_x^{-1} Q_G(n; v, x) + \delta_{v,u}, \; n \geq 0, \tag{77}$$

where the sum on the right-hand side, as indicated, is over the neighbors of u . The quantity

其中右侧的求和如标注所示，是对 u 的所有邻点求和。量

$$Q'_G(n; v, u) = \sigma_u^{-1} Q_G(n; v, u),$$

which is symmetric in v and u , then fulfills the discrete version of the diffusion equation with a source at vertex v :

关于 v 和 u 对称，满足顶点 v 处带源的扩散方程的离散形式:

$$\partial_n Q'_G(n; v, u) = -\Delta_u^G Q'_G(n; v, u) + \sigma_v^{-1}\delta_{v,u}, \; n \geq 0, \tag{78}$$

as is easily seen by subtracting $Q_G(n; v, u)$ from both sides of equation (77). Here, ∂_n denotes the difference operator with respect to "time" n , and Δ^G is the graph Laplace operator acting on functions $f : V(G) \to \mathbb{C}$ according to

将方程 (77) 两边减去 $Q_G(n; v, u)$ 后即可轻松验证该式。此处 ∂_n 表示关于"时间" n 的差分算子，Δ^G 是图拉普拉斯算子，它按如下方式作用在函数 $f : V(G) \to \mathbb{C}$ 上:

$$\Delta^G f(v) = \sigma_v^{-1} \sum_{x:(v,x)\in E(G)} (f(v) - f(x)).$$

The spectral dimension of a connected graph G is most commonly defined in terms of the decay rate of the return probability $q_G(m; v, v)$ as a function of m . More precisely, if

连通图 G 的谱维最常根据返回概率 $q_G(m; v, v)$ 随 m 变化的衰减率定义。更精确地说，若

$$q_G(m; v, v) \sim m^{-\frac{\alpha}{2}} \text{ for } m \text{ large}, \tag{79}$$

we call α the spectral dimension of G and in this case G is called recurrent if $\alpha \leq 2$, and otherwise, it is called transient. More generally, noting that $Q_G(n; v, v)$ is always an increasing function of n , the limit $Q_G(\infty; v, v) := \lim_{n\to\infty} Q_G(n; v, v)$ exists, and G is recurrent if the limit is ∞ ; otherwise, G is transient. Furthermore, we find it most convenient for our purposes to define the spectral dimension in terms of the asymptotic behavior of $Q_G(n; v, v)$ for large n . Thus, for a recurrent graph G , we set

我们称 α 为 G 的谱维，在此情形下，当 $\alpha \leq$ 小于等于 2 时 G 是常返的，否则称为暂留的。更一般地，注意到 $Q_G(n; v, v)$ 始终是 n 的增函数，极限 $Q_G(\infty; v, v) := \lim_{n\to\infty} Q_G(n; v, v)$ 存在，若该极限为 ∞ ，则 G 是常返的；否则 G 是暂留的。此外，对我们的研究目标而言，根据 $Q_G(n; v, v)$ 在 n 很大时的渐近行为定义谱维是最方便的。因此，对常返图 G ，我们设

$$d_s = 2 - 2\lim_{n\to\infty} \frac{\ln Q_G(n; v, v)}{\ln n}, \tag{80}$$

provided the limit exists (in which case its value is independent of v). The definition (80) is equivalent to (79) under mild assumptions.

前提是该极限存在 (此时极限值与 v 无关)。在温和假设下，定义 (80) 与 (79) 等价。

Note that since $1 \leq Q_G(n; v, v) \leq n$, we have $0 \leq d_s \leq 2$. Obviously, $Q_G(n; v, v)$ is not a probability, contrary to $q_G(n; v, v)$. On the other hand, letting $q_G^0(m; v, v)$ denote the first return probability after m steps of the walk, i.e.,

注意，由于 $1 \leq Q_G(n;v,v) \leq n$ ，我们可得 $0 \leq d_s \leq 2$ 。显然，与 $q_G(n;v,v)$ 不同，$Q_G(n;v,v)$ 不是概率。另一方面，设 $q_G^0(m;v,v)$ 为行走 m 步后的首次返回概率，即

$$q_G^0(m;v,v) = \sum_{\omega : v \to v, |\omega| = m} p_G(\omega),$$

we have that

我们有

$$Q_G^0(n;v,v) = \sum_{m=2}^{n} q_G^0(m;v,v)$$

is the probability that an infinite walk starting at v returns to v after at most n steps and, in particular,

是从 v 出发的无限行走至多 n 步后返回 v 的概率，特别地

$$Q_G^0(\infty;v,v) = \lim_{n\to\infty} Q_G^0(n;v,v)$$

is the probability that the infinite random walk returns at least once to v . Denoting by $\Pi_\infty^m(v)$ the set of walks that return to v at least m times, we can decompose each such walk ω into pieces $\omega^{(1)}, \ldots, \omega^{(m)}, \overline{\omega}$ such that $\omega^{(k)}; v \to v$ for $k = 1, 2, \ldots, m$, while $\overline{\omega} \in \Pi_\infty(v)$ is an end piece. Then the set $A\left(\omega^{(1)}, \ldots, \omega^{(m)}\right)$ of all paths in $\Pi_\infty(v)$ whose decomposition is of the stated form with fixed $\omega_1, \ldots, \omega_m$ and arbitrary $\overline{\omega}$ has probability

是无限随机游走至少一次返回 v 的概率。记 $\Pi_\infty^m(v)$ 为至少返回 v m 次的游走集合，我们可将每个这类游走 ω 分解为若干段 $\omega^{(1)}, \ldots, \omega^{(m)}, \overline{\omega}$ ，满足对 $k = 1, 2, \ldots, m$ 有 $\omega^{(k)}; v \to v$ ，而 $\overline{\omega} \in \Pi_\infty(v)$ 是末端段。那么，所有落在 $\Pi_\infty(v)$ 中、分解满足上述形式且固定 $\omega_1, \ldots, \omega_m$ 任意 $\overline{\omega}$ 的路径构成的集合 $A\left(\omega^{(1)}, \ldots, \omega^{(m)}\right)$ ，其概率为

$$P_{G,v}\left(A\left(\omega^{(1)}, \ldots, \omega^{(m)}\right)\right) = P_{G,v}\left(A\left(\omega^{(1)}\right)\right) \cdots P_{G,v}\left(A\left(\omega^{(m)}\right)\right).$$

By summing over $\omega^{(1)}, \ldots, \omega^{(m)}$, we obtain that the probability that the random walk returns at least m times to v equals $Q_G^0(\infty;v,v)^m$. Letting m tend to infinity we conclude that the probability that the random walk returns infinitely many times to the initial vertex vanishes if and only if $Q_G^0(\infty;v,v) < 1$ and the relation

对 $\omega^{(1)}, \ldots, \omega^{(m)}$ 求和后，我们得到随机游走至少 m 次返回 v 的概率等于 $Q_G^0(\infty;v,v)^m$ 。令 m 趋于无穷，我们可得: 随机游走无限多次返回初始顶点的概率为零当且仅当 $Q_G^0(\infty;v,v) < 1$ ，且关系

$$Q_G(\infty;v,v) = \frac{1}{1 - Q_G^0(\infty;v,v)}$$

holds. On the other hand, G is recurrent if and only if $Q_G^0(\infty;v,v) = 1$ and in that case the walk returns to the initial vertex infinitely many times with probability 1. It is well known, and easy to see, that if G is finite, then $d_s = 0$, while if G is the hypercubic lattice $\mathbb{Z}^d$ (viewed as a graph in the standard way), it is a classical result of Polya (see, for example, Chapter 2 in [17]) that G is recurrent if and only if $d \le 2$ and in all cases $d_s = d$. In this article, we are mainly concerned with recurrent graphs.

成立。另一方面，G 是常返图当且仅当 $Q_G^0(\infty; v, v) = 1$ ，此时随机游走以概率 1 无穷多次返回初始顶点。众所周知且不难验证: 若 G 是有限图，则 $d_s = 0$ 成立；若 G 是超立方格点图 $\mathbb{Z}^d$ (按标准方式视为图)，波利亚的经典结论表明 (例见文献 [17] 第 2 章)，G 是常返图当且仅当 $d \le 2$ ，且所有情况下均有 $d_s = d$ 。本文主要研究常返图。

Relation Between d_s and d_h

d_s 与 d_h 的关系

We now give an elementary proof of a well-known inequality between the spectral dimension and the Hausdorff dimension d_h valid for arbitrary recurrent graphs. This inequality has been proven under certain assumptions on the behavior of the volume of balls under scaling in [18, 19]. Related results for Riemannian manifolds were obtained earlier under similar assumptions in [20]. Here, we essentially need no assumptions beyond existence of d_s and d_h . Specifically, we now show that if G is a connected recurrent graph such that d_s and d_h both exist, then

我们现在给出谱维数与豪斯多夫维数 d_h 之间一个著名不等式的基础证明，该不等式适用于任意常返图。已有文献 [18,19] 在球体积缩放行为满足特定假设的条件下证明了该不等式，更早之前，文献 [20] 在类似假设下得到了黎曼流形的相关结果。本文中，除了 d_s 和 d_h 存在外，我们基本不需要其他额外假设。具体来说，我们现在证明: 若 G 是连通常返图，且 d_s 和 d_h 都存在，则

$$d_s \ge \frac{2d_h}{1 + d_h} \tag{81}$$

The proof is based on a simple observation whose formulation requires some further notation. Thus, let G_0 be a subgraph of a graph G . The inner boundary $\partial_{\text{in}} G_0$ of G_0 is the subgraph of G spanned by the vertices of G_0 having at least one neighbor in $V(G) \setminus V(G_0)$. Similarly, the outer boundary $\partial_{\text{out}} G_0$ is the subgraph of G spanned by the vertices not in G_0 having at least one neighbor in G_0 . The closure $\bar{G}_0$ of G_0 is the subgraph spanned by the vertices of G_0 and those of $\partial_{\text{out}} G_0$. The out-degree σ_v^{out} of a vertex v in G_0 is by definition the number of neighbors of v in G that do not belong to G_0 . In particular, a vertex of G_0 belongs to $\partial_{\text{in}} G_0$ if and only if its out-degree is positive.

证明基于一个简单观察，该观察的表述需要引入一些额外记号。设 G_0 是图 G 的子图。G_0 的内边界 $\partial_{\text{in}} G_0$ 是 G 中由至少有一个邻点在 $V(G) \setminus V(G_0)$ 的 G_0 顶点张成的子图。类似地，外边界 $\partial_{\text{out}} G_0$ 是 G 中由不在 G_0 中、且至少有一个邻点在 G_0 的顶点张成的子图。G_0 的闭包 $\bar{G}_0$ 是由 G_0 的顶点和 $\partial_{\text{out}} G_0$ 的顶点张成的子图。G_0 中顶点 v 的出度 σ_v^{out} 定义为 v 在 G 中不属于 G_0 的邻点的数量。特别地，G_0 中的顶点属于 $\partial_{\text{in}} G_0$ 当且仅当它的出度为正。

Now, let G be a connected graph and V_0 a proper subset of $V(G)$ and denote by G_0 the subgraph of G spanned by V_0 . Then, for arbitrary fixed $v_0 \in V_0$, we have

现在设 G 是连通图，V_0 是 $V(G)$ 的真子集，记 G_0 是由 V_0 张成的 G 的子图，则对任意固定的 $v_0 \in V_0$，我们有

$$\sum_{v \in \partial_{\text{in}} G_0} \sum_{\substack{\omega : v_0 \to v \\ \omega \subseteq G_0}} p_G(\omega) \sigma_v^{-1} \sigma_v^{\text{out}} \leq 1, \tag{82}$$

with equality holding if $\bar{G}_0$ is connected and recurrent.

当 $\bar{G}_0$ 连通且为常返图时等号成立。

The inequality (82) follows by observing that the left-hand side is the probability q with respect to P_{G,v_0} that a walk starting at v_0 leaves G_0. In fact, given such a walk $\widetilde{\omega}$, let v denote the last vertex in G_0 visited by $\widetilde{\omega}$ before it leaves G_0 for the first time, and let ω denote the corresponding initial part of $\widetilde{\omega}$ from v_0 to v contained in G_0. Then $v \in \partial_{\text{in}} G_0$, and there are σ_v^{out} vertices in $\bar{G}_0$ that $\widetilde{\omega}$ may hit next with each such possibility contributing a probability $p_G(\omega)\sigma_v^{-1}$ to q. This proves (82). Clearly, q only depends on $\bar{G}_0$, and if this graph is connected and recurrent, it is well known [21] that the probability for a walk to hit any given vertex of $\bar{G}_0$ equals 1. In particular, since $V_0 \neq V(G)$, it follows that $q = 1$.

不等式 (82) 可由如下观察得到: 左侧是相对于 P_{G,v_0}，从 v_0 出发的随机游走离开 G_0 的概率 q。实际上，对于这样一个游走 $\widetilde{\omega}$，设 v 是 $\widetilde{\omega}$ 第一次离开 G_0 前访问的最后一个 G_0 内顶点，ω 是 $\widetilde{\omega}$ 从 v_0 到 v、完全包含在 G_0 中的初始路径段。此时有 $v \in \partial_{\text{in}} G_0$，$\bar{G}_0$ 中存在 σ_v^{out} 个顶点可被 $\widetilde{\omega}$ 下一步访问，每种可能对 q 贡献概率 $p_G(\omega)\sigma_v^{-1}$。由此证得 (82)。显然，q 仅依赖于 $\bar{G}_0$，若该图连通且常返，根据已知结论 [21]，游走击中 $\bar{G}_0$ 任意给定顶点的概率等于 1。特别地，由 $V_0 \neq V(G)$ 可得 $q = 1$。

For use in the proof of (81), we note two useful consequences of (82). First, let G be a connected graph, and let v_0 and u_0 be two different vertices of G. Then

为了用于证明式 (81)，我们从式 (82) 得到两个有用的推论。首先，设 G 为连通图，v_0 和 u_0 是 G 的两个不同顶点。则

$$\sum_{\substack{\omega : v_0 \to v_0 \\ u_0 \in \omega}} p_G(\omega) \leq \sigma_{v_0} \sigma_{u_0}^{-1} \sum_{\omega : v_0 \to u_0} p_G(\omega), \tag{83}$$

and equality holds if G is recurrent.

当 G 常返时等号成立。

To prove this statement, we set $V_0 = V(G) \setminus \{u_0\}$, and let G_0 be the subgraph of G spanned by V_0. In other words, G_0 is obtained from G by removing u_0 and the edges containing u_0 and is frequently denoted by $G - u_0$. Now, note that $v_0 \in G_0$ and that every walk $\omega : v_0 \to v_0$ containing u_0 can be decomposed uniquely into a walk $\omega' : v_0 \to u_0$ and a walk $\omega'' : u_0 \to v_0$, such that ω'' does not return to u_0. Hence, the reverse of ω'' is a walk ω''' in G_0 from v_0 to some $v \in \partial_{\text{in}} G_0$ and one additional step to u_0. Since $\sigma_v^{\text{out}} = 1$ for all $v \in \partial_{\text{in}} G_0$ in this case,(82) gives

为证明该结论，设 $V_0 = V(G) \setminus \{u_0\}$，令 G_0 是由 V_0 张成的 G 的子图。换句话说，G_0 是通过移除 u_0 和所有包含 u_0 的边从 G 得到的，通常记作 $G - u_0$。现在注意到 $v_0 \in G_0$，且每一条包含 u_0 的路径 $\omega : v_0 \to v_0$ 都可以唯一分解为路径 $\omega' : v_0 \to u_0$ 和路径 $\omega'' : u_0 \to v_0$，其中 ω'' 不返回 u_0。因此，ω'' 的逆是 G_0 中从 v_0 到任意 $v \in \partial_{\text{in}} G_0$ 的路径 ω'''，再额外走一步到达 u_0。由于这种情况下对所有 $v \in \partial_{\text{in}} G_0$ 都有 $\sigma_v^{\text{out}} = 1$，由式 (82) 可得

$$\sum_{\substack{\omega : v_0 \to v_0 \\ u_0 \in \omega}} p_G(\omega) = \sum_{\omega' : v_0 \to u_0} p_G(\omega') \sigma_{u_0}^{-1} \sum_{v \in \partial_{\text{in}} G_0} \sum_{\substack{\omega''' : v_0 \to v \\ \omega''' \subseteq G_0}} p_G(\omega''') \sigma_v^{-1} \sigma_{v_0}$$

$$\leq \sum_{\omega' : v_0 \to u_0} \sigma_{v_0} \sigma_{u_0}^{-1} p_G(\omega')$$

with equality holding if G is recurrent. This proves (83).

当 G 常返时等号成立，由此证得式 (83)。

Second, with G and v_0 and u_0 as above, we have that

其次，在 G、v_0 和 u_0 满足上述条件时，我们有

$$\sum_{\substack{\omega : v_0 \to v_0 \\ u_0 \notin \omega}} p_G(\omega) \leq \sigma_{v_0} d_G(v_0, u_0). \tag{84}$$

In order to verify this claim, let $(v_0 v_1), (v_1 v_2), \dots, (v_{N-2} v_{N-1}), (v_{N-1} u_0)$ be a path from v_0 to w_0 of minimal length $N = d_G(v_0, u_0)$. Set $v_N = u_0$ and define k_ω, for each walk $\omega : v_0 \to v_0$, to be the maximal index k such that $v_k \in \omega$. In particular, if $u_0 \notin \omega$, then $k_\omega \leq N - 1$ and

为验证该结论，设 $(v_0 v_1), (v_1 v_2), \dots, (v_{N-2} v_{N-1}), (v_{N-1} u_0)$ 是从 v_0 到 w_0 的最短路径，长度为 $N = d_G(v_0, u_0)$。设 $v_N = u_0$，对任意路径 $\omega : v_0 \to v_0$，定义 k_ω 为满足 $v_k \in \omega$ 的最大下标 k。特别地，若 $u_0 \notin \omega$，则 $k_\omega \leq N - 1$ 且

$$v_{k_\omega} \in \omega, \; v_{k_\omega + 1}, \dots, v_N \notin \omega.$$

Given that $k_\omega = l \geq 1$, there is a unique decomposition of ω into a walk $\omega' : v_0 \to v_l$ and a walk $\omega'' : v_l \to v_0$ such that ω'' does not return to v_l. As previously, the reverse of ω'' is a walk ω''' from v_0 to a neighbor v of v_l avoiding $v_l, v_{l+1}, \dots, v_N$, and an additional last step from v to v_l. Setting $V_0 = V(G) \setminus \{v_l, \dots, v_N\}$ and $V_1 = V(G) \setminus \{v_{l+1}, \dots, v_N\}$ and letting G_0 and G_1 be the subgraphs of G spanned by V_0 and V_1, respectively, it follows that $\omega' \subseteq G_1$ and $\omega''' \subseteq G_0$. Noting that $v \in \partial_{\text{in}} G_0$ and that

给定 $k_\omega = l \geq 1$，ω 可唯一分解为路径 $\omega' : v_0 \to v_l$ 和路径 $\omega'' : v_l \to v_0$，使得 ω'' 不返回 v_l。与之前的情况相同，ω'' 的逆路径是一条从 v_0 到 v_l 的邻点 v 且避开 $v_l, v_{l+1}, \dots, v_N$ 的路径 ω'''，额外增加一步从 v 到 v_l。设定 $V_0 = V(G) \setminus \{v_l, \dots, v_N\}$ 和 $V_1 = V(G) \setminus \{v_{l+1}, \dots, v_N\}$，分别令 G_0 和 G_1 为 V_0 和 V_1 张成的 G 子图，可得 $\omega' \subseteq G_1$ 和 $\omega''' \subseteq G_0$。注意到 $v \in \partial_{\text{in}} G_0$ 且

$$p_G(\omega) = p_G(\omega') \sigma_{v_l}^{-1} \sigma_{v_0} p_G(\omega''') \sigma_v^{-1}$$

we conclude that

我们得出结论

$$\sum_{\substack{\omega:v_0\to v_0\\k_\omega=l}} p_G(\omega) \le \sigma_{v_0}\left(\sum_{\substack{\omega':v_0\to v_l\\\omega'\subseteq G_1}} p_G(\omega')\,\sigma_{v_l}^{-1}\right)\left(\sum_{\substack{v\in\partial_{\mathrm{in}}G_0\\\omega''\subseteq G_0}}\sum_{\substack{\omega''':v_0\to v\\\omega'''\subseteq G_0}} p_G(\omega''')\,\sigma_v^{-1}\right).$$

(85)

Since $v_l \in \partial_{\mathrm{in}} G_1$, it follows from (82) that the expressions in parentheses on the right-hand side are bounded by 1 such that

由于 $v_l \in \partial_{\mathrm{in}} G_1$ ，由式 (82) 可知，右侧括号中的表达式不超过 1，因此

$$\sum_{\substack{\omega:v_0\to v_0\\k_\omega=l}} p_G(\omega) \le \sigma_{v_0} \tag{86}$$

In the case $k_\omega = 0$, the walk ω'' is trivial, and the reader easily verifies that the above inequality still holds with the trivial walk also included on the left-hand side contributing 1 to the sum. Finally, summing on both sides of (86) from $l = 0$ to $l = N - 1$, the claimed inequality (84) follows.

当 $k_\omega = 0$ 时，路径 ω'' 是平凡路径，读者可轻易验证上述不等式仍然成立: 平凡路径也包含在左侧求和中，对求和贡献 1。最后，对式 (86) 两侧从 $l = 0$ 到 $l = N - 1$ 求和，即可得到所证的不等式 (84)。

We are now ready to give a proof of (81). Let $v_0 \in G$ be fixed. Since p_G is a probability distribution on walks of length m starting at v_0 , the identity

我们现在准备证明式 (81)。设 $v_0 \in G$ 固定。由于 p_G 是起点为 v_0 、长度为 m 的路径上的概率分布，等式

$$\sum_{v\in G} Q_G(n; v_0, v) = \sum_{v\in G}\sum_{\substack{\omega:v_0\to v\\|\omega|\le n}} p_G(\omega) = n+1 \tag{87}$$

holds for each $n \ge 0$. Restricting the sum on the left-hand side to vertices in $B_R(G; v_0)$, we obtain an inequality instead, which implies that there exists a vertex $v_{R,n}$ in $B_R(G; v_0)$ such that

对任意 $n \ge 0$ 成立。将左侧求和限制在 $B_R(G; v_0)$ 的顶点范围内，我们得到一个不等式，由此推出 $B_R(G; v_0)$ 中存在顶点 $v_{R,n}$ 满足

$$Q_G\left(n; v_0, v_{R,n}\right) \le \frac{n+1}{\left\|B_R(G; v_0)\right\|}, \tag{88}$$

where R is an arbitrary positive integer. Writing

其中 R 是任意正整数。记

$$Q_G(n; v_0, v_0) = \sum_{\substack{\omega: v_0 \to v_0 \\ v_{R,n} \in \omega, |\omega| \le n}} p_G(\omega) + \sum_{\substack{\omega: v_0 \to v_0 \\ v_{R,n} \notin \omega, |\omega| \le n}} p_G(\omega),$$

it follows from (83) and (84) together with (88) that

结合式 (83)、(84) 和 (88) 可得

$$\begin{aligned} Q_G(n; v_0, v_0) &\le \sigma_{v_0} \sigma_{v_{R,n}}^{-1} \sum_{\substack{\omega: v_0 \to v_{R,n} \\ |\omega| \le n}} p_G(\omega) + \sigma_{v_0} d_G(v_0, v_{R,n}) \\ &\le \sigma_{v_0} Q_G(n; v_0, v_{R,n}) + \sigma_{v_0} d_G(v_0, v_{R,n}) \\ &\le \sigma_{v_0} \left(\frac{n+1}{\left\| B_R(G; v_0) \right\|} + R \right). \end{aligned} \tag{89}$$

Now, choose R as a function of n such that the two terms in parenthesis are of the same order of magnitude. This is obtained for $R = \left\lfloor n^{\frac{1}{1+d_h}} \right\rfloor$, where $\lfloor a \rfloor$ denotes the integer part of the real number a . In this case, the inequality (89) gives

现在，将 R 取为 n 的函数，使得括号中的两项量级相同。当 $R = \left\lfloor n^{\frac{1}{1+d_h}} \right\rfloor$ 时满足该条件，其中 $\lfloor a \rfloor$ 表示实数 a 的整数部分。此时，由不等式 (89) 可得

$$\frac{\ln Q_G(n; v_0, v_0)}{\ln n} \le \frac{1}{1+d_h} + \frac{\ln \sigma_{v_0} + \ln \left(1 + \frac{n+1}{R \left\| B_R(G; v_0) \right\|} \right)}{\ln n}.$$

Here, the last term tends to zero as $n \to \infty$ by the assumption that d_h exists, and hence, by (80), we get

此处，根据 d_h 存在的假设，当 $n \to \infty$ 时最后一项趋于 0，因此由式 (80) 我们得到

$$d_s \ge 2 - 2\frac{1}{1+d_h} = \frac{2d_h}{1+d_h}$$

as desired.

证毕。

Spectral Dimension of Generic Trees

一般树的谱维

For the generic trees, we noted in section ”Hausdorff Dimension” that $d_h = 2$ with probability 1. Hence, it follows from (81) that

对于一般树，我们在“豪斯多夫维数”一节中已经指出，概率为 1 时 $d_h = 2$ 成立。因此，由式 (81) 可得

$$d_s \geq \frac{4}{3} \tag{90}$$

with probability 1, provided the limit (80) exists. We do not provide detailed arguments for the existence of the limit, but some further comments on this issue can be found at the end of this subsection. Next, we aim at proving that equality holds in (90), and for that, we need a suitable lower bound on $Q(n; v_0, v_0)$ supplementing the upper bound (89).

若极限 (80) 存在，则概率为 1 时该式成立。本文不对极限的存在性做详细论证，本小节末会对该问题做进一步补充说明。接下来我们目标证明式 (90) 取等号，为此，我们需要在式 (89) 给出的上界之外，为 $Q(n; v_0, v_0)$ 找到一个合适的下界。

In [7], a rather special type of lower bound on generating functions for return probabilities of random walk on generic trees was proven. Here, we establish a natural generalization of that bound applicable to the cumulated probabilities $Q_G(n; v_0, v_0)$ associated with an arbitrary connected graph, as stated in the following theorem:

文献 [7] 证明了一般树上随机游走返回概率生成函数的一种特殊下界。本文我们对该下界做自然推广，使其适用于任意连通图对应的累积概率 $Q_G(n; v_0, v_0)$，具体结论如下述定理:

Theorem 1. Let G be a connected graph and G_0 a finite connected subgraph of G spanned by its set of vertices V_0 . Then

定理 1 设 G 为连通图，G_0 是 G 中由顶点集 V_0 张成的有限连通子图，则有

$$Q_G(n; v_0, v_0) \geq \sigma_{v_0} \left(\frac{2|\bar{G}_0|}{n+1} + \mathcal{C}_{G,G_0}(n; v_0) \right)^{-1}, \tag{91}$$

for every vertex $v_0 \in G_0$, where $\mathcal{C}_{G,G_0}(n; v_0)$ is defined, up to a factor σ_{v_0} , as the probability for a walk starting at v_0 not to return to v_0 before leaving G_0 in at most n steps, that is

对任意顶点 $v_0 \in G_0$ 成立，其中 $\mathcal{C}_{G,G_0}(n; v_0)$ 的定义相差因子 σ_{v_0} ，具体为从 v_0 出发的游走在离开 G_0 前不晚于 n 步返回 v_0 的概率，即

$$\mathcal{C}_{G,G_0}(n; v_0) = \sigma_{v_0} \sum_{v \in \partial_{\text{in}} G_0} \sum_{\substack{\omega : v_0 \to v \\ \omega \subseteq G_0, |\omega| \leq n}} p_G(\omega) \sigma_v^{-1} \sigma_v^{\text{out}} . \tag{92}$$

Proof. We define

证明: 我们定义

$$\widetilde{Q}_G(n; v, u) = \sigma_u^{-1} \sum_{\substack{\omega : v \to u \\ \omega \subseteq G_0, |\omega| \leq n}} p_G(\omega)$$

and note that $\widetilde{Q}_G(n; v, u)$ vanishes if $v \notin V_0$ or $u \notin V_0$, while it satisfies the diffusion equation (78) for $v, u \in V_0$. Since the left-hand side of (78) is nonnegative (as $\widetilde{Q}(n; v, u)$ is a nondecreasing function of n), it follows from the maximum principle for the discrete Laplacian that $\widetilde{Q}_G(n; v_0, u)$ assumes its maximal value as a function of u at $u = v_0$, i.e.,

易知，若 $v \notin V_0$ 或 $u \notin V_0$ ，则 $\widetilde{Q}_G(n; v, u)$ 为零；而对于 $v, u \in V_0$ ，$\widetilde{Q}_G(n; v, u)$ 满足扩散方程 (78)。由于式 (78) 左侧非负 (因为 $\widetilde{Q}(n; v, u)$ 是关于 n 的非递减函数)，由离散拉普拉斯算子的最大值原理可得，$\widetilde{Q}_G(n; v_0, u)$ 作为 u 的函数，其最大值出现在 $u = v_0$ 处，即

$$\widetilde{Q}_G(n; v_0, u) \leq \widetilde{Q}_G(n; v_0, v_0), \ u \in G. \tag{93}$$

From (87), we have

由式 (87)，我们可得

$$n + 1 = \sum_{u \in G_0} \sigma_u \widetilde{Q}_G(n; v_0, u) + \sum_{u \in G} \sum_{\substack{\omega : v_0 \to u \\ \omega \notin G_0, |\omega| \leq n}} p_G(\omega). \tag{94}$$

Using (93) and that

结合式 (93) 以及

$$\sum_{u \in G_0} \sigma_u \leq 2 |\bar{G}_0|$$

the first sum on the right-hand side in (94) can be estimated from above by

式 (94) 右侧第一项可估计上界为

$$2 |\bar{G}_0| \widetilde{Q}_G(n; v_0, v_0) \ . \tag{95}$$

The last sum, on the other hand, can be estimated as follows: Any walk ω starting at v_0 which is not contained in G_0 can be decomposed in a unique way into a (possibly trivial) walk $\omega' : v_0 \to v_0$ which is contained in G_0 , followed by a walk $\omega'' : v_0 \to v$ which is likewise contained in G_0 but does not return to v_0 and such that $v \in \partial_{\text{in}} G_0$ and finally a step from v to a vertex $v' \in \partial_{\text{out}} G_0$ and a walk ω''' starting at v' . Obviously, the lengths of ω', ω'' , and ω''' sum up to at most n and $p_G(\omega) = p_G(\omega') p_G(\omega'') \sigma_v^{-1} p_G(\omega''')$. Hence, an upper bound on the last term in (94) is obtained by relaxing the constraint $|\omega'| + |\omega''| + |\omega'''| \leq n$ to $|\omega'|, |\omega''|, |\omega'''| \leq n$, in which case the sum factorizes into three terms: summation over ω' contributes a factor $\sigma_{v_0} \widetilde{Q}_G(n; v_0, v_0)$, summation over ω''' is bounded by $n + 1$ by (87), whereafter summation over ω'', v , and v' gives a factor $\sigma_{v_0}^{-1} \mathcal{C}_{G, G_0}(n; v_0)$. Hence, we have

另一方面，最后一项和可估计如下: 任意从 v_0 出发且不包含在 G_0 中的行走 ω 都可以唯一分解为: 包含在 G_0 内的 (可能平凡的) 行走 ω' ：$v_0 \to v_0$，之后接一个同样包含在 G_0 内但不返回 v_0 的行走 ω'' ：$v_0 \to v$， 满足 $v \in \partial_{\text{in}} G_0$， 最后一步从 v 走到顶点 $v' \in \partial_{\text{out}} G_0$， 再接一个从 v' 出发的行走 ω''' 。显然，ω', ω'' 和 ω''' 的长度之和至多为 n 加 $p_G(\omega) = p_G(\omega') \, p_G(\omega'') \, \sigma_v^{-1} p_G(\omega''')$。因此，通过将约束条件 $|\omega'| + |\omega''| + |\omega'''| \leq n$ 放宽为 $|\omega'|, |\omega''|, |\omega'''| \leq n$， 可得到式 (94) 最后一项的上界, 此时该和可分解为三个因子项: 对 ω' 求和贡献因子 $\sigma_{v_0} \widetilde{Q}_G(n; v_0, v_0)$，根据式 (87)，对 ω''' 求和有上界 $n+1$， 随后对 ω'', v 和 v' 求和得到因子 $\sigma_{v_0}^{-1} \mathcal{C}_{G,G_0}(n; v_0)$。因此我们得到

$$\sum_{u \in G} \sum_{\substack{\omega : v_0 \to u \\ \omega \not\subseteq G_0, |\omega| \leq n}} p_G(\omega) \leq (n+1) \, \widetilde{Q}_G(n; v_0, v_0) \, \mathcal{C}_{G,G_0}(n; v_0) \, . \tag{96}$$

Using Eqs. (94), (95), and (96), we finally arrive at

利用式 (94)、(95) 和 (96)，我们最终得到

$$n + 1 \leq \widetilde{Q}_G(n; v_0, v_0) \left\{ 2 \, |\bar{G}_0| + (n+1) \, \mathcal{C}_{G,G_0}(n; v_0) \right\},$$

which implies (91) since $\sigma_{v_0} \widetilde{Q}_G(n; v_0, v_0) \leq Q_G(n; v_0)$.

由 $\sigma_{v_0} \widetilde{Q}_G(n; v_0, v_0) \leq Q_G(n; v_0)$ 可推出式 (91) 成立。

Note that the relation of $\mathcal{C}_{G,G_0}(n; v_0)$ to an exit probability shows that it is bounded by σ_{v_0} . Clearly, $\mathcal{C}_{G,G_0}(n; v_0)$ is an increasing function of n , so the limit

注意，$\mathcal{C}_{G,G_0}(n; v_0)$ 与出逃概率的关系表明它以 σ_{v_0} 为界。显然，$\mathcal{C}_{G,G_0}(n; v_0)$ 是关于 n 的递增函数，因此极限

$$\mathcal{C}_{G,G_0}(v_0) = \lim_{n \to \infty} \mathcal{C}_{G,G_0}(n; v_0) = \sup_{n \geq 1} \mathcal{C}_{G,G_0}(n; v_0) \, , \tag{97}$$

exists and is, by definition, the effective conductance of G between v_0 and the complement of G_0 . The effective resistance of G between v_0 and the complement of G_0 is defined as

存在，根据定义，它就是 G 在 v_0 与 G_0 的补集之间的有效电导。G 在 v_0 与 G_0 的补集之间的有效电阻定义为

$$\mathcal{R}_{G,G_0}(v_0) = \left(\mathcal{C}_{G,G_0}(v_0) \right)^{-1} . \tag{98}$$

Clearly, (91) then implies

显然，式 (91) 可以推出

$$Q_G(n; v_0, v_0) \geq \sigma_{v_0} \left(\frac{2 \, |\bar{G}_0|}{n+1} + \mathcal{R}_{G,G_0}(v_0)^{-1} \right)^{-1} , \tag{99}$$

Given graphs G and G_0 as above let us define $\widehat{G}_0$ to be the graph obtained from $\bar{G}_0$ by identifying all vertices in $\partial^{\text{out}} G_0$ with a single new vertex v_1 and leaving out all edges with both endpoints in $\partial^{\text{out}} G_0$. (It should be noted that $\widehat{G}_0$ may contain multiple edges, but the reader may easily verify that all considerations in the present subsection apply with obvious modifications also to graphs with multiple edges.) It is then clear that $\mathcal{C}_{G,G_0}(v_0) = \mathcal{C}_{\widehat{G}_0,\widehat{G}_0 - v_1}(v_0)$, which is called the conductance of

给定上述的图 G 和 G_0 ，我们定义 $\widehat{G}_0$ 为由 $\bar{G}_0$ 得到的图: 将 $\partial^{\text{out}} G_0$ 中的所有顶点合并为一个新顶点 v_1 , 并去掉所有两个端点都在 $\partial^{\text{out}} G_0$ 中的边。(需要注意的是, $\widehat{G}_0$ 可能包含重边, 但读者可轻易验证, 本小节的所有讨论在稍加修改后也同样适用于含重边的图。) 那么显然，$\mathcal{C}_{G,G_0}(v_0) = \mathcal{C}_{\widehat{G}_0,\widehat{G}_0 - v_1}(v_0)$ 就是所谓的电导，即

$\widehat{G}_0$ between v_0 and v_1 . It will also be denoted by $\mathcal{C}_{\widehat{G}_0}(v_0, v_1)$. Since here $\widehat{G}_0$ can be any finite graph, this defines the conductance between any two different vertices v_0, v_1 in a finite connected graph H by

$\widehat{G}_0$ 中 v_0 与 v_1 之间的电导，也记作 $\mathcal{C}_{\widehat{G}_0}(v_0, v_1)$ 。由于此处 $\widehat{G}_0$ 可以是任意有限图，该定义即可给出有限连通图 H 中任意两个不同顶点 v_0, v_1 之间的电导，定义式为

$$\mathcal{C}_H(v_0, v_1) = \sum_{\omega : v_0 \leftrightarrow v_1} \sigma_{v_0} p_H(\omega), \tag{100}$$

where we use the notation $\omega : v_0 \leftrightarrow v_1$ to denote a path from v_0 to v_1 that does not hit the end-vertices at intermediate steps. Clearly, $\mathcal{C}_H(v_0, v_1)$ is symmetric in v_0 and v_1 .

其中我们用记号 $\omega : v_0 \leftrightarrow v_1$ 表示从 v_0 到 v_1 、且中间步骤不经过端点的路径。显然 $\mathcal{C}_H(v_0, v_1)$ 关于 v_0 和 v_1 对称。

Recalling (82) and noting that any path $\omega : v_0 \to v_1$ can uniquely be decomposed into a path $\omega' : v_0 \to v_0$ (possibly trivial) not hitting v_1 and a path $\omega'' : v_0 \leftrightarrow v_1$, we get that

回顾式 (82)，注意到任意路径 $\omega : v_0 \to v_1$ 都可以唯一分解为一条不经过 v_1 的路径 $\omega' : v_0 \to v_0$ (可能是平凡路径) 与路径 $\omega'' : v_0 \leftrightarrow v_1$ ，因此我们得到

$$\mathcal{C}_H(v_0, v_1) \sum_{\substack{\omega' : v_0 \to v_0 \\ v_1 \notin \omega}} p(\omega') \sigma_{v_0}^{-1} = 1,$$

which implies that the resistance $\mathcal{R}_H(v_0, v_1) := (\mathcal{C}_H(v_0, v_1))^{-1}$ can be expressed as

这说明电阻 $\mathcal{R}_H(v_0, v_1) := (\mathcal{C}_H(v_0, v_1))^{-1}$ 可以表示为

$$\mathcal{R}_H(v_0, v_1) = \sum_{\substack{\omega : v_0 \to v_0 \\ v_1 \notin \omega}} p(\omega) \sigma_{v_0}^{-1}. \tag{101}$$

The relation of these definitions to the physical notion of conductance and resistance in electrical networks is perhaps not obvious at this stage. From (100), it is clear that conductance and resistance of a single edge are equal to 1, and it is simple to verify, using (100) and (101), that the standard laws for composing resistances in a series or in parallel hold. We refer to Chapter 2 of [17] for a more general and detailed account of these aspects, including Rayleigh's monotonicity principle which states that the effective resistance is a nondecreasing function of the edge resistances. In our case of unit edge resistances, this principle implies that contracting an edge in H that does not connect v_0 and v_1 , i.e., deleting the edge and identifying its end vertices, reduces the resistance $\mathcal{R}_H(v_0, v_1)$ or leaves it unchanged.

这些定义与电网中电导和电阻的物理概念之间的关系目前可能并不直观。由式 (100) 可知，单条边的电导和电阻都等于 1；且利用式 (100) 和 (101) 很容易验证，电阻串联、并联的标准组合定律均成立。关于这些内容，包括瑞利单调性原理 (该原理指出有效电阻是边电阻的非减函数)，更全面详细的阐述可参见文献 [17] 的第二章。在我们的单位边电阻情形下，该原理说明: 收缩 H 中不连接 v_0 和 v_1 的边 (即删除该边并将其两个端点重合) 会使电阻 $\mathcal{R}_H(v_0, v_1)$ 减小或保持不变。

We remark that the inequality (84) can now be rewritten as

我们注意到不等式 (84) 现在可以改写为

$$\mathcal{R}_H(v_0, v_1) \leq d_H(v_0, v_1) \tag{102}$$

for any pair of different vertices v_0, v_1 in a finite connected graph H . Moreover, we have that equality holds if H is a tree, i.e.,

对有限连通图 H 中任意一对不同顶点 v_0, v_1 成立。此外，当 H 是树时等号成立，即

$$\mathcal{R}_T(v_0, v_1) = d_T(v_0, v_1) \quad \text{if } T \text{ is a finite tree.} \tag{103}$$

Indeed, if $d_T(v_0, v_1) = 1$, only one path consisting of the edge connecting v_0 and v_1 contributes on the right-hand side of (100) and gives 1 . If $d_T(v_0, v_1) = k \geq 2$, let v be a vertex in the interior of the unique path connecting v_0 and v_1 , and let T_1 be the sub-tree of T spanned by v and its descendants when considering v_0 as the root of T , and let T_0 be the tree spanned by the remaining vertices and v . Then T_0 and T_1 only share the single vertex v , and hence by the law of coupling resistances in a series, we have

事实上，若 $d_T(v_0, v_1) = 1$ ，则仅有一条由连接 v_0 和 v_1 的边构成的路径对 (100) 的右侧有贡献，结果为 1。若 $d_T(v_0, v_1) = k \geq 2$ ，设 v 是连接 v_0 和 v_1 的唯一路径内部的顶点，将 v_0 视为 T 的根，设 T_1 是由 v 及其后代在 T 中张成的子树，再设 T_0 是由剩余顶点和 v 张成的树。此时 T_0 与 T_1 仅共享顶点 v ，因此根据串联耦合电阻定律，我们有

$$\mathcal{R}_T(v_0, v_1) = \mathcal{R}_{T_0}(v_0, v) + \mathcal{R}_{T_1}(v, v_1).$$

The claim (103) now follows trivially by induction.

命题 (103) 可由归纳法轻易得证。

Given a rooted graph G , let $\mathcal{R}(R)$ denote the resistance between the root v_0 and the complement of the ball $B_R(G)$ of radius R around the root, and assume that for some $\kappa \geq 0$ it holds for R large that

给定根图 G ，设 $\mathcal{R}(R)$ 表示根节点 v_0 与根周围半径为 R 的球 $B_R(G)$ 的补集之间的电阻，假设对某个 $\kappa \geq 0$ ，当 R 足够大时，下式成立

$$\mathcal{R}(r) \geq \text{ const. } R^{\kappa} .$$

Note that (102) implies the constraint

注意到 (102) 蕴含约束条件

$$\kappa \leq 1 . \tag{104}$$

If the Hausdorff dimension d_h of G exists, we obtain, by choosing $G_0 = B_R(G)$ in (91) where $R = \left\lfloor n^{\frac{1}{d_h+\kappa}} \right\rfloor$, that

若 G 的豪斯多夫维数 d_h 存在，我们在 $R = \left\lfloor n^{\frac{1}{d_h+\kappa}} \right\rfloor$ 成立时，在 (91) 中选取 $G_0 = B_R(G)$ ，可得

$$Q_G(n; v_0, v_0) \geq \text{ const. } n^{\frac{\kappa}{d_h+\kappa}}$$

and, hence,

因此可得

$$d_s \leq \frac{2d_h}{d_h + \kappa}, \tag{105}$$

thus supplementing the lower bound (81).

从而补充了下界 (81)。

If $G_0^K, K = 1, 2, 3, \ldots$, is a sequence of finite connected graphs as in Theorem 1 containing a fixed vertex v_0 and such that the graph distance from v_0 to $\partial_{\text{in}} G_0^K$ tends to infinity as $K \to \infty$, then $\sigma_{v_0}^{-1} \lim_{K\to\infty} \mathcal{C}_{G,G_0^K}(v_0)$ exists and equals the escape probability from v_0 , which is the probability for an infinite walk starting at v_0 never to return to v_0 . Hence, by the discussion of recurrency in section ”Definition of Spectral Dimension of Recurrent Graphs,” we conclude that this quantity vanishes exactly if G is recurrent. In particular, we get that if G is recurrent, then $\mathcal{C}_{G,G_0^K}(n; v_0)$ converges to 0 uniformly in n as $K \to \infty$. In order to exploit Theorem 1, we need more detailed information on the decay rate of $\mathcal{C}_{G,G_0^K}(n; v_0)$ or $\mathcal{C}_{G,G_0^K}(v_0)$ for an appropriate choice of G_0^K . This is a nontrivial problem for general graphs, but if G is a tree we can make use of (103) as will be seen.

若 $G_0^K, K=1,2,3,\ldots$ 是如定理 1 所述的有限连通图序列，其中包含固定顶点 v_0，且从 v_0 到 $\partial_{\text{in}} G_0^K$ 的图距离随 $K\to\infty$ 趋于无穷，则 $\sigma_{v_0}^{-1}\lim_{K\to\infty}\mathcal{C}_{G,G_0^K}(v_0)$ 存在，且等于从 v_0 出发的逃逸概率，也就是从 v_0 出发的无穷随机游走永远不返回 v_0 的概率。因此，结合"循环图的谱维数定义"一节中对循环性的讨论，我们可得该量当且仅当 G 是循环图时为零。特别地，我们得到若 G 是循环图，则 $\mathcal{C}_{G,G_0^K}(n;v_0)$ 随 $K\to\infty$ 在 n 上一致收敛到 0。为了利用定理 1，我们需要关于 $\mathcal{C}_{G,G_0^K}(n;v_0)$ 或 $\mathcal{C}_{G,G_0^K}(v_0)$ 在 G_0^K 的合适选取下的衰减率的更详细信息。这对一般图来说是一个非平凡问题，但当 G 是树时，我们可以如后文所示利用 (103) 解决该问题。

We are now in a position to apply the previous results to the case of generic random trees and prove the desired upper bound on their spectral dimension. Recalling the one-spine character of the generic trees, we let T be such a tree and aim at applying Theorem 1 with G_0^K equal to the sub-tree spanned by the vertices of the (finite) branches rooted at the spine vertices $s_1,\ldots,s_K$.

现在我们可以将此前的结果应用于一般随机树的情形，证明其谱维数的期望上界。回顾一般树的单脊椎性质，设 T 为这样一棵树，我们目标应用定理 1，令 G_0^K 为根节点在脊椎顶点 $s_1,\ldots,s_K$ 上的 (有限) 分支顶点张成的子树。

Denoting as previously by Br_i the union of the branches rooted at a fixed spine vertex s_i , we then have

和前文一样，记 Br_i 为根在固定脊椎顶点 s_i 上的所有分支的并，我们可得

$$|\bar{G}_0^K| = \sum_{i=1}^{K}|Br_i| + K + 1 \tag{106}$$

and obviously, $|\bar{G}_0^K|\geq|B_K|$. We can now use (21) to estimate the growth rate of $|G_0^K|$ by first establishing the following lemma.

显然有 $|\bar{G}_0^K|\geq|B_K|$。我们现在可以利用 (21) 估计 $|G_0^K|$ 的增长率，首先证明下述引理。

Lemma 1. There exist constants $\bar{c}>0$ and $u_0>0$ such that for all $K\geq 1$ and all $u>u_0$ the following inequality holds:

引理 1. 存在常数 $\bar{c}>0$ 和 $u_0>0$，使得对所有 $K\geq 1$ 和所有 $u>u_0$，都有如下不等式成立:

$$v\left(\{T : K^{-2}|\bar{G}_0^K|\geq u\}\right)\leq\frac{\bar{c}}{\sqrt{u}}. \tag{107}$$

Proof. By (106), it clearly suffices to show (107) with $X_K=\sum_{i=1}^{K}|Br_i|$ replacing $|\bar{G}_0^K|$. We take as starting point the following inequality which can be found, e.g., in [22, Section 8.7]:

证明. 根据 (106)，显然只需证明用 $X_K=\sum_{i=1}^{K}|Br_i|$ 替换 $|\bar{G}_0^K|$ 后的 (107)。我们以如下不等式为起点，该不等式可在例如文献 [22, 第 8.7 节] 中找到:

$$v\left(\{X_K \geq u\}\right) \leq \alpha u \int_0^{\frac{1}{u}} \left(1 - \operatorname{Re} \bar{Z}\left(e^{iv}\right)^K\right) dv, \tag{108}$$

where α is a universal constant and $\bar{Z}\left(e^{iv}\right)^K$ is the characteristic function of X_K as a sum of K independent and identically distributed (i.i.d.) random variables. From (21), we have

其中 α 是一个通用常数，$\bar{Z}\left(e^{iv}\right)^K$ 是 X_K 作为 K 个独立同分布 (i.i.d.) 随机变量之和的特征函数。由 (21)，我们可得

$$\bar{Z}\left(e^{iv}\right) = e^{\bar{c}_0\sqrt{1-e^{iv}}+O\left(\left|1-e^{iv}\right|\right)}$$

and hence, for $v > 0$ sufficiently small,

因此，当 $v > 0$ 足够小时，

$$\begin{aligned}\operatorname{Re} \bar{Z}\left(e^{iv}\right)^K &= \operatorname{Re} e^{K\bar{c}_0\sqrt{1-e^{iv}}+KO\left(\left|1-e^{iv}\right|\right)} \\ &= \operatorname{Re} e^{K\bar{c}_0\sqrt{-iv+O(v^2)}+KO(v)} \\ &= \operatorname{Re} e^{\frac{1}{\sqrt{2}}\bar{c}_0K(1-i)\sqrt{v}\sqrt{1+O(v)}+KO(v)} \\ &= e^{\frac{1}{\sqrt{2}}\bar{c}_0K\sqrt{v}+KO(v)} \cos\left(\frac{1}{\sqrt{2}}\bar{c}_0K\sqrt{v} + KO\left(v\right)\right).\end{aligned}$$

By Taylor expanding the exponential and cosine functions, it follows that there exists a $\delta > 0$ such that

对指数函数和余弦函数做泰勒展开，可得存在 $\delta > 0$ 使得

$$1 - \operatorname{Re} \bar{Z}\left(e^{iv}\right)^K = \frac{1}{\sqrt{2}}\bar{c}_0K\sqrt{v} + O\left(K^2v\right) \leq \bar{c}_0K\sqrt{v}, \text{ for } 0 \leq K\sqrt{v} \leq \delta.$$

Using this estimate in (108), we get

将该估计代入 (108)，我们得到

$$v\left(\{X_K \geq u\}\right) \leq \frac{2\alpha\bar{c}_0K}{3\sqrt{u}}$$

if $\frac{K}{\sqrt{u}} \leq \delta$. Upon replacing u by uK^2, the claimed inequality follows for $u \geq \delta^{-2}$ with $\bar{c} = \frac{2}{3}\alpha\bar{c}_0$.

若 $\frac{K}{\sqrt{u}} \leq \delta$。将 u 替换为 uK^2，对带有 $\bar{c} = \frac{2}{3}\alpha\bar{c}_0$ 的 $u \geq \delta^{-2}$ 即可得到所断言的不等式。

Setting $K = 2^M$ and $u = M^3$ in (107), we get

在 (107) 中令 $K = 2^M$ 和 $u = M^3$ ，我们得到

$$\sum_{M=1}^{\infty} v\left(\left\{T : 4^{-M}\left|\bar{G}_0{}^{2^M}\right| \geq M^3\right\}\right) < \infty.$$

Hence, by the Borel-Cantelli lemma, we conclude that with probability 1, it holds that

因此，由博雷尔-坎泰利引理，我们得出结论: 该式以概率 1 成立

$$\left|\bar{G}_0^{2^M}\right| \leq M^3 4^M \tag{109}$$

for $M > M_0$ where M_0 is an integer depending on T . Furthermore, given an arbitrary $K \geq 1$, we can choose M such that $2^{M-1} \leq K \leq 2^M$ and conclude that

对 $M > M_0$ 成立，其中 M_0 是依赖于 T 的整数。此外，给定任意 $K \geq 1$ ，我们可以选取 M 使得 $2^{M-1} \leq K \leq 2^M$ ，并得出结论

$$\left|\bar{G}_0^K\right| \leq \left|\bar{G}_0^{2^M}\right| \leq M^3 4^M \leq 4K^2\left(\frac{\ln K}{\ln 2} + 1\right)^3 \leq c_1 K^2 (\ln K)^3 \tag{110}$$

for $K > K_0$, where $c_1 > 0$ is a numerical constant independent of T , while K_0 may depend on T .

对 $K > K_0$ 成立，其中 $c_1 > 0$ 是不依赖于 T 的数值常数，而 K_0 可以依赖于 T 。

For a fixed $T \in \mathcal{T}_\infty$ as above, let $n \geq 1$ be given and let $K = \left\lfloor n^{\frac{1}{3}} \right\rfloor$, and assume n is large enough so that $K > K_0$. It then follows from (91) and (110) that

对于如上所述的固定 $T \in \mathcal{T}_\infty$ ，设给定 $n \geq 1$ 并令 $K = \left\lfloor n^{\frac{1}{3}} \right\rfloor$ ，假设 n 足够大使得 $K > K_0$ 。那么由 (91) 和 (110) 可得

$$Q_T(n; s_0, s_0) \geq Q_T\left(K^3; s_0, s_0\right) \geq \left(\frac{c_1 K^2 (\ln K)^3}{K^3 + 1} + K^{-1}\right)^{-1} \geq C_1 (\ln n)^{-3} n^{\frac{1}{3}},$$

(111)

where $C_1 > 0$ is a constant, and we have also used that $\mathcal{C}_{T,G_0^K}(s_0) = (K+1)^{-1}$ by (103). Using (80), the definition of d_s , it follows that $d_s \leq \frac{4}{3}$ with probability 1 .

其中 $C_1 > 0$ 为常数，我们还利用了由 (103) 得到的 $\mathcal{C}_{T,G_0^K}(s_0) = (K+1)^{-1}$ 。根据 d_s 的定义，利用 (80) 可推出 $d_s \leq \frac{4}{3}$ 以概率 1 成立。

For the sake of completeness, we mention that by elaborating on the upper bound (89) on $Q_T(n; s_0, s_0)$ in a similar way as above and using the lower bound on

为求完整，我们在此说明: 沿用上述类似方法细化 $Q_T(n; s_0, s_0)$ 的上界 (89)，并结合

$|B_R(T)|$ in (72), we obtain an upper bound on $Q_T(n; s_0, s_0)$ analogous to (111). More precisely, one gets that

(72) 中 $|B_R(T)|$ 的下界，我们可以得到与 (111) 类似的 $Q_T(n; s_0, s_0)$ 上界。更准确地说，可得

$$C_1(\ln n)^{-3} n^{\frac{1}{3}} \leq Q_T(n; s_0, s_0) \leq C_2(\ln n)^2 n^{\frac{1}{3}}$$

for a suitable constant $C_2 > 0$, which of course also implies $d_s = \frac{4}{3}$.

存在合适的常数 $C_2 > 0$ ，这当然也能推出 $d_s = \frac{4}{3}$ 。

Spectral Dimension of Causal Triangulations

因果三角剖分的谱维数

In this section, we show that the spectral dimension of the CDT ensemble defined in section "Causal Triangulations" equals 2. Thus, even though the Hausdorff dimensions of the CDTs and of the corresponding generic tree are identical, the spectral dimensions are not. This is due to the higher connectivity of the CT which leads to different behavior of the resistance between the root and the complement of the ball around the root for large radius.

在本节中，我们证明“因果三角剖分”一节中定义的 CDT 系综的谱维数等于 2。因此，尽管 CDT 和对应一般树的豪斯多夫维数相同，二者的谱维数却并不相等。这是因为 CT 的连通性更高，使得大半径下根节点与根节点球补集之间的电阻表现出不同的行为。

To obtain the upper bound $d_s \leq 2$, we make use of an argument that is most easily understood in terms of resistance estimates. Considering an arbitrary infinite causal triangulation G , let G_R be the graph obtained from the ball $B_R(G)$ by collapsing its boundary to a single vertex u (i.e., by contracting all the boundary edges). Then the resistance of G between the central vertex S_0 and the complement of the ball $B_R(G)$ is identical to the resistance of G_R between S_0 and u . By contracting the edges in each $S_r, r = 1, \ldots, R$, we obtain, by the Rayleigh’s monotonicity principle, a network with lower resistance between S_0 and u . On the other hand, this network is a series of resistances $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_R$, where $\mathcal{R}_k$ is the resistance of $\Delta\left(\sum_k\right)$ unit resistances connected in parallel. Hence,

为了得到上界 $d_s \leq 2$ ，我们采用了一个最容易通过电阻估计理解的论证。任取一个无限因果三角剖分 G ，记 G_R 是将球 $B_R(G)$ 的边界收缩为单个顶点 u (即收缩所有边界边) 得到的图。此时 G 中中心顶点 S_0 与球 $B_R(G)$ 补集之间的电阻，等于 G_R 中 S_0 与 u 之间的电阻。根据瑞利单调性原理，对每个 $S_r, r = 1, \ldots, R$ 中的边进行收缩后，我们得到的网络在 S_0 和 u 之间的电阻更低。另一方面，该网络是由电阻 $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_R$ 串联而成，其中 $\mathcal{R}_k$ 是 $\Delta\left(\sum_k\right)$ 个单位电阻并联后的总电阻。因此，

$$\mathcal{R}_{G,B_R}(S_0) \geq \sum_{k=0}^{R} \frac{1}{\Delta\left(\sum_k\right)}. \tag{112}$$

In order to make use of (99), we need a suitable lower bound on the resistance $\mathcal{R}_{G,B_R}(S_0)$. In view of (112), we therefore want to estimate the probability $\rho(\{G:\ \Delta(\sum_k) > K)$. For this purpose, we use (16) to write

为了利用 (99) 式，我们需要对电阻 $\mathcal{R}_{G,B_R}(S_0)$ 给出一个合适的下界。根据 (112) 式，我们需要估计概率 $\rho(\{G:\Delta(\sum_k) > K)$。为此，我们利用 (16) 式写出

$$\rho(\{G:\ B_R(G) = B_R(G_0)\}) = D_R(G_0)\, 2^{D_R(G_0)+1} 2^{-\left\|B_R(G_0)\right\|} \tag{113}$$

for any fixed infinite causal triangulation $G_0 \in C$, where it is used that $Z_0 = \frac{1}{2}$ and $\zeta_0 = \frac{1}{4}$ in (16) for the uniform planar tree and also $\left\|B_R(G_0)\right\| = |B_{R+1}(\beta(G_0))|$. Since the number of causal triangulations of the annulus $\sum_k$ with l_k vertices on the inner boundary, among which one is marked, and l_{k+1} vertices on the outer boundary equals $\binom{l_k + l_{k+1} - 1}{l_k - 1}$, it follows by a straightforward combinatorial argument that

对任意固定的无限因果三角剖分 $G_0 \in C$ 成立，这里我们用到了均匀平面树在 (16) 式中的 $Z_0 = \frac{1}{2}$、$\zeta_0 = \frac{1}{4}$，以及 $\left\|B_R(G_0)\right\| = |B_{R+1}(\beta(G_0))|$。由于环带 $\sum_k$ 满足: 内边界有 l_k 个顶点 (其中一个顶点被标记)，外边界有 l_{k+1} 个顶点，其因果三角剖分的数量为 $\binom{l_k + l_{k+1} - 1}{l_k - 1}$，因此通过简单的组合论证可得

$$\rho(\{G:\ D_R = l_R, D_{R+1} = l_{R+1}\}) = l_{R+1} 2^{-l_{R+1}} \binom{l_R + l_{R+1} - 1}{l_R - 1} \frac{1}{R(R-1)} \left(\frac{R-1}{2R}\right)^{l_R}$$

(114)

(see [16] for details). Summing this identity over $l_R + l_{R+1} > K$ then yields

(详见文献 [16])。对 $l_R + l_{R+1} > K$ 求和该恒等式后可得

$$\rho\left(\left\{G:\ \Delta\left(\sum\nolimits_R\right) > K\right\}\right) = \frac{K + 2R - 1}{2R - 1}\left(1 - \frac{1}{2R}\right)^K. \tag{115}$$

Now, let $a > 2$ be fixed and define

现在，固定 $a > 2$ 并定义

$$\mathcal{A}_R = \left\{G:\ \Delta\left(\sum\nolimits_R\right) > aR\ln R\right\}.$$

Then (115) implies that $\rho(\mathcal{A}_R) \leq (1 + a\ln R) R^{-a}$, and hence

此时 (115) 式推出 $\rho(\mathcal{A}_R) \leq (1 + a\ln R) R^{-a}$，因此

$$\sum_{R=1}^{\infty} \rho(\mathcal{A}_R) < \infty$$

By the Borel-Cantelli lemma, it follows that with probability 1 it holds that

根据博雷尔-坎泰利引理，可得该式以概率 1 成立

$$\Delta\left(\sum_{R}\right) \leq aR\ln R$$

for R large enough. Consequently, for all such $G \in \mathcal{C}_{\infty}$, we have

当 R 足够大时。因此，对所有这类 $G \in \mathcal{C}_{\infty}$ ，我们有

$$\mathcal{R}_{G,B_R}(S_0) \leq \text{ const. } \sum_{k=1}^{R}(ak\ln k)^{-1} \leq \text{ const. } \ln\ln R. \tag{116}$$

Setting $R = \left\lfloor n^{\frac{1}{3}} \right\rfloor$ in (99), it follows from the bound (110) and the previous estimate that

将 $R = \left\lfloor n^{\frac{1}{3}} \right\rfloor$ 代入 (99) 式，结合 (110) 式的界和上述估计可得

$$Q_G(n;S_0,S_0) \geq \left(\frac{c_1R^2(\ln R)^3}{n+1} + (a\ln\ln R)^{-1}\right)^{-1} \geq C_1'\ln\ln n \tag{117}$$

for a suitable constant $C_1' > 0$ (depending on G). By the definition of d_s , this evidently implies the desired upper bound $d_s \leq 2$.

对于某个合适的常数 $C_1' > 0$ (取决于 G)。由 d_s 的定义，这显然能推导出我们想要的上界 $d_s \leq 2$ 。

To obtain a useful upper bound on $Q_G(n;S_0,S_0)$, the inequality (89) is not applicable since it does not capture the dependence of d_s on the behavior of the resistance between the root and the complement of balls at large radii. In our particular case, however, the bound

若要得到 $Q_G(n;S_0,S_0)$ 的有效上界，不等式 (89) 并不适用，因为它没有体现出 d_s 对大半径下根与球补集之间电阻性质的依赖关系。但在我们的特例中，该下界

$$Q_G(n;S_0) \leq \sigma_{S_0}\mathcal{R}_{G,B_R}(S_0) \quad \text{for } n \leq R, \tag{118}$$

which follows immediately from the definitions of Q_G and $\mathcal{R}_{G,B_R}(S_0)$, is sufficient, provided a suitable upper bound on the resistance can be obtained. In [23], it is shown by an argument requiring rather advanced probabilistic techniques that the bound

可直接由 Q_G 和 $\mathcal{R}_{G,B_R}(S_0)$ 的定义得到，只要能得到电阻的合适上界，它就足够用了。文献 [23] 中通过一个需要相当高深概率技术的论证证明了，该界

$$\mathcal{R}_{G,B_R}(S_0) \leq e^{\text{const.}\sqrt{\ln R}}$$

holds for R sufficiently large with probability 1 . Hence, it follows by setting $R = n^b$ in (118), where $b \geq 1$, that with probability 1 we have

对于足够大的 R 以概率 1 成立。因此，对 (118) 代入 $R = n^b$ (其中 $b \geq 1$) 后可得，我们以概率 1 有

$$Q_G(n; S_0) \leq \sigma_{S_0} e^{\text{const.} \sqrt{\ln n}}$$

for n large enough. Clearly, this implies the lower bound $d_s \geq 2$.

对于足够大的 n 成立。显然，这可以推导出下界 $d_s \geq 2$ 。

Curvature and Matter Fields on the CDT

CDT 上的曲率与物质场

Modifications of the graph weights, by introducing terms that bias the vertex degree or adding extra degrees of freedom (often referred to as ”matter fields”) to the graphs, might lead to different critical behavior. Some of these modifications can be analyzed using the bijection β , but understanding of others remains seriously incomplete. Some examples are discussed here in the grand canonical ensemble framework.

通过引入偏向顶点度的项，或向图中额外添加自由度 (通常称为“物质场”) 修改图权重，可能产生不同的临界行为。其中部分修改可借助双射 β 分析，但对其余修改的理解仍严重不足。本文将在巨正则系综框架下讨论若干示例。

Curvature

曲率

The simplest elaboration of the graph weights is to introduce an extra factor $q^{\sigma_v - 6}$ for each vertex into the definition of W_M (49). This is the analogue of including the Ricci scalar curvature term from the Einstein action in a continuum gravity theory. It is trivial at fixed topology in two dimensions because $\sum_{v \in G} (\sigma_v - 6)$ is proportional to the Euler characteristic of G so W_M is simply multiplied by a constant.

对图权重最简单的拓展是在 W_M (式 49) 的定义中为每个顶点引入额外因子 $q^{\sigma_v - 6}$ 。这对应于连续引力理论中，在爱因斯坦作用量里加入里奇标量曲率项。在二维固定拓扑下这是平凡的，因为 $\sum_{v \in G} (\sigma_v - 6)$ 与 G 的欧拉特征成正比，因此只需要给 W_M 整体乘一个常数。

A different higher curvature term was proposed by [12]. Recalling the definition of σ_{bv} and σ_{fv} from section ”Definition,” introduce the extra factor

文献 [12] 提出了另一种高阶曲率项。回顾“定义”小节中 σ_{bv} 和 σ_{fv} 的定义，我们引入如下额外因子

$$Q(p, q) = \prod_{v \in G \setminus S_0} q^{\frac{1}{2}|\sigma_{fv} - 2|} p^{\frac{1}{2}|\sigma_{bv} - 2|} \tag{119}$$

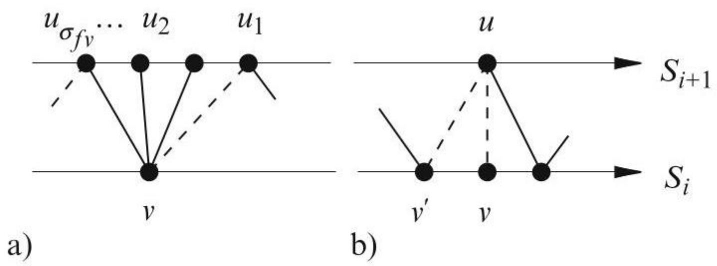
into the graph weight for the disk partition function (49). Decreasing q, p from 1 enhances the weight for vertices which belong to three triangles each in the forward and backward direction and thus introduces a bias toward graphs that, at least locally, are regular triangulations of the plane. (It was argued in [24] that this is equivalent to the naive discretization of an extrinsic curvature squared term in the continuum gravity action.) However, it is straightforward to see that the critical properties of W_M [16] remain unchanged as follows. Consider a vertex $v \in S_i(G)$ which shares edges with $u_m \in S_{i+1}(G), m = 1, \dots \sigma_{fv}$ (see Fig. 7a). Then, making use of the bijection β , the q -dependent part of (119) is given by

> 加入圆盘配分函数的图权重 (式 49) 中。当 q, p 从 1 减小时，前后方向各属于三个三角形的顶点权重会被提高，因此会对图产生偏向性——更倾向于 (至少局部是) 平面正则三角剖分的图。(文献 [24] 指出，这等价于连续引力作用量中外曲率平方项的朴素离散化。) 但我们可以直接看出，W_M 的临界性质 [16] 保持不变，推导如下: 考虑一个顶点 $v \in S_i(G)$ ，它与 $u_m \in S_{i+1}(G), m = 1, \dots \sigma_{fv}$ 共享边 (参见图 7a)。利用双射 β ，式 (119) 中依赖 q 的部分可写为

$$\prod_{v \in G} q^{\frac{1}{2}|\sigma_{fv}-2|} = \prod_{v \in \beta(G):\sigma_{fv} \geq 2} q^{\frac{1}{2}(\sigma_{fv}-2)} \prod_{u \in \beta(G):\sigma_{fu}=1} q^{\frac{1}{2}}. \tag{120}$$

Fig. 7 The neighborhood of a vertex $v \in G$ when (a) $\sigma_{fv} > 1$ and (b) $\sigma_{fv} = 1$. Dashed lines represent edges that are present in G but not in the tree $T = \beta(G)$

> 图 7 (a) $\sigma_{fv} > 1$ 和 (b) $\sigma_{fv} = 1$ 情况下顶点 $v \in G$ 的邻域。虚线表示存在于 G 中但不存在于树 $T = \beta(G)$ 中的边



The p -dependent part of (119) can be rewritten to give

> 式 (119) 中依赖 p 的部分可以重写为

$$\prod_{v \in G \setminus S_0} p^{\frac{1}{2}|\sigma_{bv}-2|} = \prod_{v \in G \setminus S_0:\sigma_{bv}>1} p^{\frac{1}{2}(\sigma_{bv}-2)} \prod_{u \in G \setminus S_0:\sigma_{bu}=1} p^{\frac{1}{2}}. \tag{121}$$

Vertices $u \in G$ with $\sigma_{bu} = 1$ cannot be the leftmost descendant of a vertex $v \in \beta(G)$ (see Fig. 7a). So each vertex $v \in \beta(G)$ with $\sigma_{fv} \geq 2$ is associated with precisely $\sigma_{fv} - 2$ vertices $u \in G$ with $\sigma_{bu} = 1$ which gives

> 带有 $\sigma_{bu} = 1$ 的顶点 $u \in G$ 不可能是顶点 $v \in \beta(G)$ 的最左后代 (参见图 7a)。因此, 每个带有 $\sigma_{fv} \geq 2$ 的顶点 $v \in \beta(G)$ 恰好对应 $\sigma_{fv} - 2$ 个带有 $\sigma_{bu} = 1$ 的顶点 $u \in G$ ，由此可得

$$\prod_{u\in G\setminus S_0:\sigma_{bu}=1} p^{\frac{1}{2}} = \prod_{u\in\beta(G):\sigma_{fu}\geq 2} p^{\frac{1}{2}(\sigma_{fu}-2)}. \tag{122}$$

Finally, vertices $u\in S_{i+1}(G)$ with $\sigma_{bu}>1$ must be the leftmost descendant of a vertex $v\in\beta(G)$ (see Fig. 7b); each vertex $v'\in S_i(G)$ with $\sigma_{fv'}=1$ then increments σ_{bu} by one, so,

最后，满足 $\sigma_{bu}>1$ 的顶点 $u\in S_{i+1}(G)$ 一定是顶点 $v\in\beta(G)$ 的最左后代 (参见图 7b)；因此每个满足 $\sigma_{fv'}=1$ 的顶点 $v'\in S_i(G)$ 都会让 σ_{bu} 加 1，于是有

$$\prod_{v\in G\setminus S_0:\sigma_{bv}>1} p^{\frac{1}{2}(\sigma_{bv}-2)} = \prod_{u\in\beta(G):\sigma_{fu}=1} p^{\frac{1}{2}}. \tag{123}$$

Combining (119), (120), (121), (122), and (123) gives

合并式 (119)、(120)、(121)、(122) 和 (123) 可得

$$Q(p,q) = \prod_{v\in\beta(G):\sigma_{fv}\geq 2} (pq)^{\frac{1}{2}(\sigma_{fv}-2)} \prod_{u\in\beta(G):\sigma_{fu}=1} (pq)^{\frac{1}{2}}, \tag{124}$$

so that each leaf in $T=\beta(G)$ gets a weight $(pq)^{\frac{1}{2}}$ and all other vertices a weight $(pq)^{\frac{1}{2}(k_v-1)}$, where k_v is the number of descendants.

因此 $T=\beta(G)$ 中的每个叶节点得到权重 $(pq)^{\frac{1}{2}}$ ，所有其他顶点得到权重 $(pq)^{\frac{1}{2}(k_v-1)}$ ，其中 k_v 是后代的数量。

Without loss of generality, set $p=q$; the recurrence (55) is then replaced by

不失一般性，令 $p=q$ ；此时递推式 (55) 替换为

$$\begin{aligned}\widetilde{w}_h(g,q,z) &= q + \sum_{k=1}^{\infty} g^{2k} q^{k-1} (\widetilde{w}_{h-1}(g,q,z))^k \\ &= \frac{q + g^2(1-q^2)\widetilde{w}_{h-1}(g,q,z)}{1-g^2 q\widetilde{w}_{h-1}(g,q,z)}.\end{aligned} \tag{125}$$

It is easy to show (by direct substitution in (125) and using (55)) that the solution satisfying $\widetilde{w}_1(g,q,z) = zg^{-1}$ is

不难证明 (通过直接代入式 (125) 并利用式 (55))，满足 $\widetilde{w}_1(g,q,z)=zg^{-1}$ 的解为

$$\widetilde{w}_h(g,z) = q - q^{-1} + \frac{q^{-1}}{1+g^2(1-q^2)} w_h\left(\frac{g}{1+g^2(1-q^2)}, qz + g(1-q^2)\right).$$

(126)

It follows from (59) that the functions $\widetilde{w}_h(g,z)$ for every h are analytic in the region

由式 (59) 可得，对任意 h ，函数 $\widetilde{w}_h(g,z)$ 在该区域内都是解析的

$$A : |g| < (1+q)^{-1}, |z| < 1, \tag{127}$$

so this modification has no effect on the large graph behavior, and the model is in the same universality class for all q . It is clearly possible to define curvature-dependent weights that take a form different from (119), for example, $q^{|\sigma_{fv}+\sigma_{bv}-4|}$ or $q^{(\sigma_{fv}+\sigma_{bv}-4)^2}$; but nothing is known about such systems.

因此这一修正对大图形行为没有影响，对于所有 q ，该模型都属于同一个普适类。显然我们可以定义形式不同于式 (119) 的曲率相关权重，例如 $q^{|\sigma_{fv}+\sigma_{bv}-4|}$ 或 $q^{(\sigma_{fv}+\sigma_{bv}-4)^2}$ ；但目前对这类体系尚无任何研究结论。

Dimers

二聚物

Dimer models on fixed lattices exhibit a rich structure. In particular, they have a critical point, at which the dimers condense, whose scaling limit is related to the Lee-Yang singularity and is described by a conformal field theory (CFT) [25]. The model of dimers coupled to CTs, defined below, can be solved by a bijection, which is a generalization of β , to labeled tree [26-29]. There are new phases in which the interaction between the dimers and the graphs becomes strong and changes the universality class from the pure CT case of section ”Disk and Annulus Partition Functions.” The steps to demonstrate this are outlined here; full details are in [29].

固定晶格上的二聚物模型呈现出丰富的结构。尤其是，这类模型存在一个临界点: 二聚物会在该临界点发生凝聚，其标度极限与李-杨奇点相关，可由共形场论 (CFT) 描述 [25]。下文定义的耦合因果三角剖分 (CT) 的二聚物模型可通过双射求解，该双射是 β 对标记树的推广 [26-29]。当二聚物与图的相互作用变强时，会出现新的相，改变普适类，使之不同于“圆盘与环形配分函数”一节中的纯因果三角剖分情形。本文概述了证明该结论的步骤；完整细节参见文献 [29]。

Dimers are objects that are dual to edges and may be assigned freely subject to the dimer rule that each triangle is allowed to contain only one dimer. The possible types of dimer on a CT, and the vertices which ”own” them, are shown in Fig. 8; we will assume that there are no dimers allowed on the edges that are attached to the central vertex S_0 or that enter the boundary triangles of $G \in \mathcal{C}^{(h)}$. We denote the dimer configurations allowed by the dimer rule for $G \in \mathcal{C}$ by $\mathcal{L}(G)$ and the number of times a dimer of type i appears in a dimer configuration $l \in \mathcal{L}(G)$ by l_i . The disk partition function for this model is then defined by assigning each dimer of type i a weight ξ_i and is given by

二聚物是与边对偶的对象，可自由赋值，但需满足二聚物规则: 每个三角形仅允许容纳一个二聚物。因果三角剖分上二聚物的可能类型，以及“拥有”它们的顶点如图 8 所示；我们规定，连接到中心顶点 S_0 的边上，或是进入 $G \in \mathcal{C}^{(h)}$ 边界三角形的边上，不允许存在二聚物。我们用 $\mathcal{L}(G)$ 表示 $G \in \mathcal{C}$ 中满足二聚物规则的允许二聚物构型，用 l_i 表示类型 i 的二聚物在二聚物构型 $l \in \mathcal{L}(G)$ 中出现的次数。该模型的圆盘配分函数定义为: 给每个类型 i 的二聚物赋予权重 ξ_i ，形式为

$$W_M(g, \{\xi\}, z; h) = \sum_{G \in \mathcal{C}^{(h)}} \sum_{l \in \mathcal{L}(G)} |S_{h-1}(G)| (z/g)^{|S_{h-1}(G)|} g^{\Delta(G)} \xi_1^{l_1} \xi_2^{l_2} \xi_{2'}^{l_{2'}} \xi_3^{l_3} \xi_4^{l_4}.$$

(128)

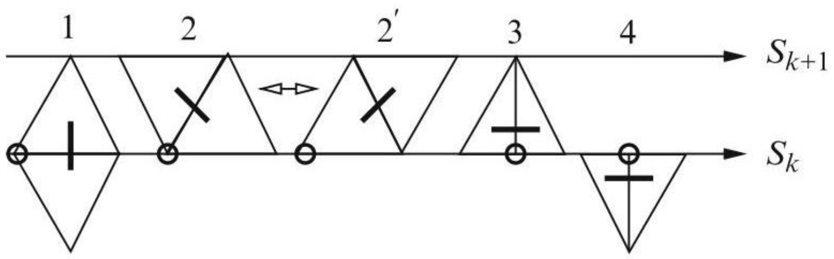


Fig. 8 The possible types of dimer on a CT, together with their index; the vertex that owns the dimer in each case is marked with a circle. The double arrow indicates the equivalence of types 2 and 2′ in W_M

图 8 因果三角剖分上二聚物的可能类型及其索引；每种情况中拥有该二聚物的顶点用圆圈标出。双箭头表示类型 2 与 2′ 在 W_M 中等价

A crucial simplification [26] is provided by the observation that for every graph-and-dimer configuration with a type 2′ dimer there is another which differs only by having a type 2 dimer on a flipped edge (see Fig. 8). It follows that

文献 [26] 给出了一个关键简化: 观察发现，对任意包含 2′ 型二聚物的图-二聚物构型，都存在另一个构型，二者的区别仅在于翻转边上存在一个 2 型二聚物 (参见图 8)。由此可得

$$W_M(g, \xi_1, \xi_2, \xi_{2'}, \xi_3, \xi_4, z; h) = W_M(g, \xi_1, \xi_2 + \xi_{2'}, 0, \xi_3, \xi_4, z; h), \tag{129}$$

so $\xi_{2'}$ can be set to zero, and the sum over dimer configurations limited to those with no type 2′ dimers. It is convenient also to let type 0 mean no dimer and then define $\xi = \{\xi_0 = 1, \xi_1, \xi_2, \xi_3, \xi_4\}$. We see, by applying the dimer rule, that each vertex v can own at most one of type 0, 1, 2, and 3 dimers and that if it owns a type 0, 2, or 3 dimer, it can also own a type 4, but that the combination of type 1 and type 4 is forbidden. So we assign to v a two-component label ℓ_v, denoting the dimers it owns, which can take values

因此 $\xi_{2'}$ 可设为零，我们只需对不包含 2′ 型二聚物的二聚物构型求和。我们也可以方便地令 0 型表示不存在二聚物，再定义 $\xi = \{\xi_0 = 1, \xi_1, \xi_2, \xi_3, \xi_4\}$。应用二聚物规则后可以看到，每个顶点 v 最多可拥有一个 0 型、1 型、2 型或 3 型二聚物；如果顶点拥有 0 型、2 型或 3 型二聚物，它还可以再拥有一个 4 型二聚物，但 1 型和 4 型的组合是被禁止的。因此我们给 v 分配一个双分量标签 ℓ_v 来标记它拥有的二聚物，标签可以取值

$$\ell_v = (p_v, q_v) \in \{0, 1, 2, 3\} \times \{0, 4\} \setminus \{(1, 4)\}. \tag{130}$$

The dimer rule then implies a set of constraints $\mathcal{N}$ on the allowed labels, ℓ_v and $\ell_{v'}$, of neighboring vertices, v and v', in $G \in C$.

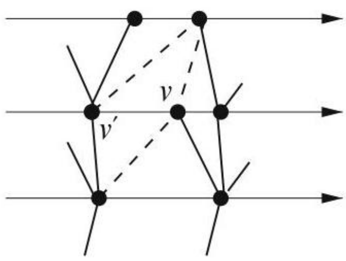
二聚物规则随后对 $G \in C$ 中相邻顶点 v 和 v' 允许的标签 ℓ_v 和 $\ell_{v'}$ 给出了一组约束 $\mathcal{N}$。

Since each $v \in G$ is also a vertex of the corresponding tree $T = \beta(G)$, the labels ℓ_v can also be associated with the vertices of T. Then $\mathcal{N}$ induces a set of rules for the labeling of $v \in T$ [29]. These rules are local in T with one exception which is illustrated in Fig. 9. Two vertices which are nearest neighbors in G can be arbitrarily widely separated in T ; we deal with this nonlocal constraint in T by forbidding $p_v = 3$ if v is the most anticlockwise descendant of another vertex. The outcome is a model of dimers on CT that is equivalent to a model of labeled trees with a set of local constraints $\widetilde{\mathcal{N}}$ on the labels; this model can in turn be solved by generalizing the methods of section ”Grand Canonical Ensemble and the Scaling Limit.”

由于每个 $v \in G$ 同时也是对应树 $T = \beta(G)$ 的顶点，因此标记 ℓ_v 也可关联到 T 的顶点上。随后 $\mathcal{N}$ 会引出一组 $v \in T$ 的标记规则 [29]。除了一种例外情况，这些规则在 T 上都是定域的，图 9 展示了该例外。G 中的两个相邻顶点在 T 中可以任意远离；我们在 T 中处理该非定域约束的方式是: 若 v 是另一顶点的最逆时针方向后代，则禁止 $p_v = 3$ 。最终得到的 CT 二聚体模型等价于带一组定域约束 $\widetilde{\mathcal{N}}$ 的标记树模型；该模型可以通过推广“巨正则系综与标度极限”一节的方法求解。

Fig. 9 Dashed lines represent edges that are present in G but not in the tree $T = \beta(G)$. If $p_{v'} = 3$, then $p_v = 1$ is forbidden, which is a nonlocal constraint on T

<id=”1”> 图 9 虚线表示存在于 G 中但不存在于树 $T = \beta(G)$ 中的边。若 $p_{v'} = 3$ ，则 $p_v = 1$ 被禁止，这是对 T 的一个非定域约束



To solve this model by the decomposition used in section ”Grand Canonical Ensemble and the Scaling Limit,” we have to keep track of the label ℓ assigned to the vertex adjacent to the root of the tree T (the root itself has no label). We then denote the allowed label configurations by $\mathcal{L}^{\ell}(T)$ and define the corresponding partition function

为了用“巨正则系综与标度极限”一节的分解方法求解该模型，我们需要记录分配给树 T 根邻顶点的标记 ℓ (根本身没有标记)。我们将允许的标记构型记为 $\mathcal{L}^{\ell}(T)$ ，并定义相应的配分函数

$$w_h^{\ell}(g,\xi,z) = \sum_{h' \le h} \sum_{T \in \mathcal{T}^{(h')}} \sum_{l \in \mathcal{L}^{\ell}(T)} (z/g)^{|S_h(T)|} \left(\prod_{v \in T \setminus r} g^{2(\sigma_v - 1)} \xi_{p_v} \xi_{q_v} \right). \tag{131}$$

It is easy to show, by the arguments used in section ”Disk and Annulus Partition Functions,” that the disk partition function is given by

<id=”3”> 利用“圆盘与环带配分函数”一节的论证，很容易证明圆盘配分函数可表示为

$$W_M(g,\xi,z;h) = g^2 z \frac{\partial}{\partial z} w_h^{(0,0)}(g,\xi,z). \tag{132}$$

The trees contributing at height h to (131) can be decomposed into trees of height $h-1$ as shown in Fig. 10. The local nature of the labeling rules ensures that it is only necessary to keep track of the labels at the first vertex of the component trees so the right-hand side is still made up of geometric series, albeit more complicated than for the case without dimers (55). The relationships obtained by resumming these series can be reduced to just two for $w_h^{(0,0)}$ and $w_h^{(2,0)}$ which take the form

对高度为 h 、贡献至 (131) 式的树，可以将其分解为高度为 $h-1$ 的树，如图 10 所示。标记规则的定域性保证我们只需记录分量树首个顶点的标记，因此等式右侧仍为几何级数，只是比无 dimer 的情况 (55) 更复杂。重求和这些级数后得到的关系可简化为仅关于 $w_h^{(0,0)}$ 和 $w_h^{(2,0)}$ 的两个方程，形式如下

$$w_{h+1}^{(0,0)} = F^{(1)}\left(w_h^{(0,0)}, w_h^{(2,0)}, g, \xi\right),\ w_{h+1}^{(2,0)} = F^{(2)}\left(w_h^{(0,0)}, w_h^{(2,0)}, g, \xi\right),$$

(133)
where

<id=”5”> 其中

$$F^{(1)}\left(f_1, f_2, g, \xi\right) = \frac{1}{1-B},$$

$$F^{(2)}\left(f_1, f_2, g, \xi\right) = \frac{\xi_2}{1-B}\left(\frac{g^2 f_1 + g^2 f_2 + g^4 \xi_3 f_1}{1 - g^4 \xi_3 \xi_4 f_1}\right),$$

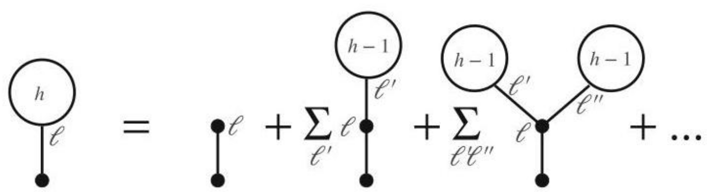
$$B = g^2 \xi_1 f_1 + g^4 \xi_4 \xi_3 f_1 + \frac{g^2\left(g^2 \xi_3 f_1 + f_1 + f_2\right)\left(1 + g^2 \xi_4 f_1 + g^2 \xi_4 f_2\right)}{1 - g^4 \xi_3 \xi_4 f_1}.$$

(134)Iterating these relations with initial data $w_1^{\ell}(g, \xi, z)$ (which are easily computed) will give the partition functions for any finite height h .

利用初始数据 $w_1^{\ell}(g, \xi, z)$ (这些数据很容易计算) 迭代这些关系，就可以得到任意有限高度 h 的配分函数。

Fig. 10 The decomposition of trees leading to (133). The sums over labels are constrained to satisfy the labeling rules $\widetilde{\mathcal{N}}$

<id=”7”> 图 10 导出 (133) 式的树分解。对标记的求和受标记规则 $\widetilde{\mathcal{N}}$ 约束



The grand canonical partition functions unconstrained by height are given by

<id=”8”> 不受高度约束的巨正则配分函数可表示为

$$w_{\infty}^{\ell}(g,\xi)=\sum_{T\in\mathcal{T}}\sum_{l\in\mathcal{L}^{\ell}(T)}\left(\prod_{v\in T\smallsetminus r}g^{2(\sigma_v-1)}\xi_{p_v}\xi_{q_v}\right). \tag{135}$$

They are simply the fixed points of the recurrence equations so satisfy (133) with $w_{h+1}^{\ell}=w_h^{\ell}=w_{\infty}^{\ell}$ for $\ell=(0,0),(2,0)$. These equations reduce to a quartic equation for $w_{\infty}^{(0,0)}$ which can in principle be solved exactly to determine the partition functions, although the explicit form of the solutions is not very useful and we will proceed in a different way. The solutions are also the limits, if these exist, of the sequences $\mathcal{S}^{\ell}=\left(w_h^{\ell},h=1,2,\ldots\right)$.

它们就是递推方程的不动点，因此满足 (133) 式，其中 $w_{h+1}^{\ell}=w_h^{\ell}=w_{\infty}^{\ell}$ 替换为 $\ell=(0,0),(2,0)$。这些方程可约化为一个关于 $w_{\infty}^{(0,0)}$ 的四次方程，原则上可以精确求解得到配分函数，但解的显式形式没什么用，因此我们将采用其他方法。这些解同时也是序列 $\mathcal{S}^{\ell}=\left(w_h^{\ell},h=1,2,\ldots\right)$ 的极限 (若序列收敛)。

As usual, we expect that the functions $w_h^{\ell}(g,\xi,z)$ for every h and the function $w_h^{\infty}(g,\xi)$ are analytic in some region

和通常情况一样，我们预期对任意 h，函数 $w_h^{\ell}(g,\xi,z)$ 以及函数 $w_h^{\infty}(g,\xi)$ 在某个区域内都是解析的

$$A_{\xi}:|g|<g_c(\xi),|z|<z_c(\xi). \tag{136}$$

However, this depends upon the sequences S_{ℓ} being smooth and converging to w_{∞}^{ℓ}. If $\xi_i\geq 0,\forall i$, then $F^{(1)},F^{(2)}$ are positive convex functions of their first two arguments, and the properties of the system are basically the same as for the pure CT model; the presence of the dimers does not drive any change in the geometrical properties, and there is no long-range cooperative behavior of the dimers themselves. If the ξ_i are sufficiently negative, then convexity is no longer guaranteed, so this behavior can change. Indeed, if they are too negative, then the sequences $\mathcal{S}^{\ell}$ become oscillatory, and the model has no meaning in the statistical mechanical sense. New and interesting physics emerges for an intermediate set of dimer weights Ξ which separate the CT-like region from the oscillatory region.

然而，这依赖于序列 S_{ℓ} 光滑且收敛于 w_{∞}^{ℓ}。若 $\xi_i\geq 0,\forall i$，则 $F^{(1)},F^{(2)}$ 是其前两个自变量的正凸函数，系统性质与纯 CT 模型基本一致；二聚体的存在不会改变几何性质，二聚体本身也不存在长程合作行为。若 ξ_i 足够小 (负得足够多)，则无法保证凸性，因此行为会发生改变。实际上，如果 ξ_i 负得过多，序列 $\mathcal{S}^{\ell}$ 会变为振荡序列，该模型在统计力学意义上不成立。当二聚体权重 Ξ 处于中间区间时，会涌现出新颖有趣的物理，这个区间分隔了类 CT 区域和振荡区域。

We proceed by analyzing the behavior in the vicinity of $g=g_c(\xi)$ assuming the convergence properties discussed above. Adopting the simplified notation $w_{\infty}^1\equiv w_{\infty}^{(0,0)}(g,\xi)$ and $w_{\infty}^2\equiv w_{\infty}^{(2,0)}(g,\xi)$, we have

我们基于上述讨论的收敛性质，继续分析 $g=g_c(\xi)$ 附近的行为。采用简化记号 $w_{\infty}^1\equiv w_{\infty}^{(0,0)}(g,\xi)$ 和 $w_{\infty}^2\equiv w_{\infty}^{(2,0)}(g,\xi)$，我们得到

$$w_{\infty}^k=F^{(k)}\left(w_{\infty}^1,w_{\infty}^2,g,\xi\right),\ k=1,2, \tag{137}$$

and

和

$$\frac{\partial w_\infty^k}{\partial g} = \left((1-\mathbb{T})^{-1}\right)^{kl} \frac{\partial}{\partial g} F^{(l)}\left(w_\infty^1, w_\infty^2, g, \xi\right), \tag{138}$$

where

其中

$$\mathbb{T}^{kl} = \frac{\partial}{\partial w_\infty^l} F^{(k)}\left(w_\infty^1, w_\infty^2, g, \xi\right) \equiv F_l^{(k)}. \tag{139}$$

At $g = 0, \mathbb{T}$ vanishes; as $g \uparrow g_c(\xi), w_\infty^k$ develops nonanalytic behavior when the largest eigenvalue of $\mathbb{T}$ reaches one. We will denote by w_c^k the value of w_∞^k at $g_c(\xi)$, by $(\lambda_1 = 1, \lambda_2)$ the eigenvalues at criticality of $\mathbb{T} = \mathbb{T}_c$, by $u_{1,2}$ the corresponding eigenvectors, and by $\bar{u}_{1,2}$ vectors orthogonal to $u_{1,2}$, respectively. $\mathbb{T}_c$ is not symmetric, and it can be shown that if the second eigenvalue $\lambda_2 = 1$, then for some values of ξ, it has Jordan normal form where u_1 is a regular eigenvector and $\mathbb{T}_c u_2 = u_2 + \varepsilon(\xi) u_1$.

在 $g = 0, \mathbb{T}$ 处该项为零；当 $g \uparrow g_c(\xi), w_\infty^k$ 在 $\mathbb{T}$ 的最大特征值达到 1 时发展出非解析行为。我们将临界态下 $g_c(\xi)$ 处 w_∞^k 的值记为 w_c^k，将 $\mathbb{T} = \mathbb{T}_c$ 在临界态的特征值记为 $(\lambda_1 = 1, \lambda_2)$，将对应的特征向量记为 $u_{1,2}$，分别将正交于 $u_{1,2}$ 的向量记为 $\bar{u}_{1,2}$。$\mathbb{T}_c$ 不是对称矩阵，可以证明当第二特征值 $\lambda_2 = 1$ 时，对于某些 ξ 取值，它具有若尔当标准型，其中 u_1 是正则特征向量，$\mathbb{T}_c u_2 = u_2 + \varepsilon(\xi) u_1$.

Expanding (137) about the critical point by setting

通过设定如下参数在临界点附近展开式 (137)

$$w_\infty^i = w_c^i + \phi^i$$

$$g = g_c(\xi) - \Delta g \tag{140}$$

we obtain

我们得到

$$(1-\mathbb{T})^{ij} \phi^j = -\Delta g \left(\frac{\partial F^{(i)}}{\partial g} + \frac{\partial \mathbb{T}^{ij}}{\partial g} \phi^j\right) + \frac{1}{2} F_{lk}^{(i)} \phi^l \phi^k$$

$$+\frac{1}{3!} F_{klm}^{(i)} \phi^k \phi^l \phi^m + O\left((\Delta g)^2, \phi^4\right). \tag{141}$$

The different phases of the model arise when ξ is tuned so that particular combinations of the coefficients in (141) vanish. With no constraints on these coefficients, we obtain the generic case whose behavior is the same as the pure CT model,

当调节 ξ 使得式 (141) 中特定系数组合为零时，就会出现模型的不同相。若对这些系数无约束，我们得到一般情形，其行为和纯 CT 模型一致,

$$\phi^i = -\phi_c(\xi)(\Delta g)^{\frac{1}{2}} u_1^i + O(\Delta g), \tag{142}$$

where $\phi_c(\xi) > 0$. Imposing the constraint

其中 $\phi_c(\xi) > 0$ 。施加约束

$$\bar{u}_2^i \left(u_1^l F_{lk}^{(i)} u_1^k \right) = 0, \tag{143}$$

leads to the Tri-critical phase in which

会得到三临界相，其中

$$\phi^i = -\phi_c(\xi)(\Delta g)^{\frac{1}{3}} u_1^i + O\left((\Delta g)^{\frac{2}{3}} \right). \tag{144}$$

Imposing the additional constraint

施加额外约束

$$\bar{u}_{2c}^i \frac{\partial F^{(i)}}{\partial g} = 0, \tag{145}$$

leads to the Dense Dimer phase. In this case, ϕ_i behaves as in (142), but other properties are different as we discuss next. The constraints (143) and (145) are conditions on the dimer weights ξ ; the first defines the region Ξ introduced above.

会得到稠密二聚体相。这种情况下，ϕ_i 的行为和式 (142) 一致，但其他性质不同，我们接下来讨论。约束 (143) 和 (145) 是对二聚体权重 ξ 的条件；第一个约束定义了上文引入的区域 Ξ 。

The unconstrained disk partition functions (135) can be written in the form

无约束圆盘配分函数 (135) 可以写为如下形式

$$w_\infty^\ell(g,\xi) = \sum_{N=1}^{\infty} w_N^\ell(\xi) g^{2N}. \tag{146}$$

It follows by general considerations from (142) and (144) that

由式 (142) 和 (144) 经一般推导可得

$$\lim_{N\to\infty} \frac{\log w_N^\ell(\xi)}{N} = -\log g_c(\xi) \tag{147}$$

therefore $\mu(\xi) = -\log g_c(\xi)$ is the thermodynamic free energy density, and the dimer density is then defined by $\chi(\xi) = -\xi_j \partial_{\xi_j} \log g_c(\xi)$. In the pure CT phase $\chi(\xi)$ is an analytic function, but as ξ approaches $\xi_c \in \Xi$ it develops nonanalytic behavior,

因此 $\mu(\xi) = -\log g_c(\xi)$ 是热力学自由能密度，二聚体密度随后由 $\chi(\xi) = -\xi_j \partial_{\xi_j} \log g_c(\xi)$ 定义。在纯 CT 相中，$\chi(\xi)$ 是一个解析函数，但当 ξ 趋近于 $\xi_c \in \Xi$ 时，它出现非解析行为，

$$\chi(\xi) = R_1(\xi) + R_2(\xi) |\xi - \xi_c|^{\sigma}, \tag{148}$$

where $R_{1,2}$ are regular functions and σ is usually called the dimer (density) exponent. If ξ_c is in the Tri-critical phase, $\sigma = \frac{1}{2}$ and χ itself remains finite, but its derivative diverges; on the other hand, if ξ_c is in the Dense Dimer phase, then $\sigma = -\frac{1}{3}$, χ diverges at $\xi = \xi_c$ and the dimers condense, hence the name.

其中 $R_{1,2}$ 是正则函数，σ 通常被称为二聚体 (密度) 指数。若 ξ_c 处于三临界相，$\sigma = \frac{1}{2}$ 和 χ 本身保持有限，但其导数发散；另一方面，若 ξ_c 处于致密二聚体相，则 $\sigma = -\frac{1}{3}, \chi$ 在 $\xi = \xi_c$ 处发散，二聚体发生凝聚，因此得名。

For each of the new phases, the local Hausdorff dimension d_h and the scaling amplitude can be calculated. Although these phases only occur when dimer weights are negative and individual graph weights can certainly be negative, it turns out that the theory at $g = g_c$ is nonetheless described by a bijection to a labeled single spine tree [28]. Thus, formally, at the level of expectation values, it is possible to repeat the considerations of section "Hausdorff Dimension;" when $\lambda_2 < 1$, we find that $d_h = 1$, which is not very interesting from the physical point of view. However, there is still some freedom in the choice of ξ which can be adjusted to impose the condition that $\mathbb{T}_c$ is not diagonalizable but has a Jordan normal form; this causes the finite trees attached to the spine in a typical graph to become more bushy, and for both the Tri-critical and Dense Dimer phases, it can be shown that

对于每一个新相，都可以计算局部豪斯多夫维数 d_h 和标度振幅。尽管这些相仅在二聚体权重为负时出现，且单个图权重也确实可以为负，但结果表明，$g = g_c$ 处的理论仍可通过双射对应到一个带标记的单脊树 [28]。因此，形式上，在期望值层面，可以重复“Hausdorff Dimension”一节中的讨论；当 $\lambda_2 < 1$ 时，我们得到 $d_h = 1$ ，这从物理角度来看并不十分有趣。不过，在选择 ξ 时仍有一定自由度，可以调整它来满足以下条件: $\mathbb{T}_c$ 不可对角化，但具有若尔当标准形；这使得典型图中附着在脊上的有限树变得更加茂密，并且对于三临界相和致密二聚体相，可以证明:

$$\langle B_R \rangle = \text{ const. } R^3 + O\left(R^2\right), \tag{149}$$

so the local Hausdorff dimension $d_h = 3$. Unlike the pure CT case of section "Disk and Annulus Partition Functions," the disk partition functions at finite h cannot be computed in closed form, but their structure is very similar. Formally, the scaling amplitudes are defined by setting $g = g_c(\xi) - \theta^{d_H}, y = y_c(\xi)\left(1 - Y\theta\Lambda^{-d_H^{-1}}\right)$, $h = H\theta^{-1}\Lambda^{-d_H^{-1}}$ and taking $\theta \to 0$,

因此局部豪斯多夫维数 $d_h = 3$ 。与“Disk and Annulus Partition Functions”一节中的纯 CT 情况不同，有限 h 处的圆盘配分函数无法得到闭合形式解，但其结构非常相似。形式上，标度振幅通过令 $g = g_c(\xi) - \theta^{d_H}, y = y_c(\xi)\left(1 - Y\theta\Lambda^{-d_H^{-1}}\right)$、$h = H\theta^{-1}\Lambda^{-d_H^{-1}}$ 并取 $\theta \to 0$ 来定义，

$$W_M^s(\Lambda, \xi, Y; H) = \lim_{\theta \to 0} W_M\left(g_c(\xi) - \theta^{d_H}, y_c(\xi)\left(1 - \theta Y\Lambda^{-d_H^{-1}}\right), H\theta^{-1}\Lambda^{d_H^{-1}}\right).$$

(150)

In practice, they can be computed in the scaling limit where the finite difference equations (133) become solvable differential equations. In the Tri-critical phase, $d_H = 3$ and

实际上，它们可以在标度极限中计算，此时差分方程 (133) 变为可解的微分方程。在三临界相中，$d_H = 3$ 且

$$W_M^s(\Lambda, Y, H) = \text{ const. } \frac{\sum_i \alpha_i e^{-\alpha_i \Lambda^{\frac{1}{3}} H}}{\left(\sum_i \left(\alpha_i^2 \Lambda^{\frac{1}{3}} + \alpha_i Y\right) e^{\alpha_i \Lambda^{\frac{1}{3}} H}\right)^2}, \tag{151}$$

where the sum runs over the three cube roots of unity, $\alpha_i, i = 1, 2, 3$. While this amplitude has an exponential decay envelope similar to the CT case (61), it has oscillations superimposed. This reflects the negative dimer weights; taking the weights more negative destroys the convergence of the sequences S^ℓ so the scaling limit no longer exists. In the Dense Dimer phase, $d_H = 2$ and

其中求和遍历三次单位根，$\alpha_i, i = 1, 2, 3$ 。尽管该振幅具有与式 (61) 中 CT 情况类似的指数衰减包络，但其上叠加了振荡。这反映了二聚体权重为负的特性；权重越负，序列 S^ℓ 的收敛性就越差，标度极限因此不再存在。在致密二聚体相中，$d_H = 2$ 且

$$W_M^s(\Lambda, Y, H) = \text{ const. } \frac{\Lambda}{\left(\Lambda^{\frac{1}{2}} \sinh H\Lambda^{\frac{1}{2}} + Y \cosh H\Lambda^{\frac{1}{2}}\right)^2}. \tag{152}$$

From the physical point of view, this is the most interesting phase. At large H , the behavior of W_M^s deviates from the pure CT case by exponentially damped terms, as if there is some kind of matter interacting weakly with gravity. This is consistent with what is known numerically about other matter degrees of freedom (albeit with only positive weights) interacting with CT, which we discuss next.

从物理角度来看，这是最有趣的一个相。在大 H 处，W_M^s 的行为偏离了纯 CT 情况，出现了指数阻尼项，仿佛存在某种物质与引力发生弱相互作用。这与其他物质自由度 (即便仅权重为正的情况) 和 CT 相互作用的数值已知结论一致，我们接下来会对此进行讨论。

Ising Spins

伊辛自旋

The Ising model on a fixed two-dimensional square lattice was first solved by Onsager in 1944. It is very well known to exhibit a second-order phase transition between a disordered phase at weak coupling (high temperature) and an ordered phase at strong coupling (low temperature); the scaling limit at the critical coupling is a conformal field theory containing a single Majorana fermion. In contrast, no method to solve the model of Ising spins coupled to CTs is known. Numerical simulations [30] indicate that the effect of the spins on the geometry is weak, and vice versa, so that the critical exponents do not change from the Onsager values and $d_H = 2$; this is corroborated by weak coupling expansions [31] and is also true of the generalization with the three-state Potts model coupled to CT [32]. There are only a few rigorous results which we now discuss briefly.

固定二维正方格子上的伊辛模型最早由昂萨格在 1944 年求解。众所周知，该模型存在二级相变: 弱耦合 (高温) 下为无序相, 强耦合 (低温) 下为有序相; 临界耦合下的标度极限是一个包含单个马约拉纳费米子的共形场论。相比之下，目前尚无方法求解耦合因果三角剖分的伊辛自旋模型。数值模拟 [30] 表明, 自旋对几何的影响很弱, 反之亦然, 因此临界指数与昂萨格给出的数值没有变化, $d_H = 2$ ；弱耦合展开 [31] 也证实了这一点，耦合因果三角剖分的三态 Potts 模型推广情形同样符合该结论 [32]。目前只有少量严格结果，我们在此做简要讨论。

For Ising spins on a fixed CT, $G \in \mathcal{C}$, drawn from the ensemble with the measure ρ (48) , the existence of a non-magnetized single phase at weak coupling, and multiple phases at strong coupling, was proved in [9]. It was also proved that the critical coupling is almost surely independent of G for such quenched systems.

对于取自测度为 ρ (48) 的系综的固定因果三角剖分上的伊辛自旋，$G \in \mathcal{C}$ ，文献 [9] 证明了弱耦合下存在非磁化单相，强耦合下存在多相。该文献还证明，对于这类淬火系统，临界耦合几乎必然与 G 无关。

These estimates make extensive use of the tree correspondence. However, they do not easily extend to the CDT case, because then G has to be chosen with a measure $\rho' \neq \rho$ that includes the effect of the spin partition function in the graph weight. This annealed model was studied in [33] where an upper bound on the radius of convergence, expressed as a function of the spin coupling strength, of the grand canonical ensemble was found. It was also shown that at weak coupling the magnetization of the spin at the vertex S_0 on the disk vanishes. Similar results for the annealed q -state Potts model coupled to CTs were obtained in [34]. There is still no proof known to us of the existence of multiple phases at strong coupling in the annealed models.

这些估计大量用到了树对应关系，但它们不容易推广到 CDT 情形，因为在 CDT 中 G 必须由包含图权重中自旋配分函数效应的测度 $\rho' \neq \rho$ 选取。这种退火模型已在文献 [33] 中得到研究，该研究得到了巨正则系综收敛半径的上界，该上界表示为自旋耦合强度的函数。研究还表明，弱耦合下，圆盘上顶点 S_0 处的自旋磁化强度为零。文献 [34] 得到了耦合因果三角剖分的退火 q 态 Potts 模型的类似结果。目前我们尚未发现退火模型强耦合下多相存在的相关证明。

Where Next?

下一步研究方向?

As we have seen through this article, there are several outstanding questions concerning CDT and its extensions with extra degrees of freedom coupled to the geometry, which we draw together here. Firstly, the relationship between the grand canonical ensemble, where only expectation values can be computed, and the infinite graph ensemble, where the graphs dominating the measure have many properties in common, is unclear. The ”observable” quantities naturally defined in one differ subtly from those naturally defined in the other. This relationship could be established by the construction of a limiting distribution of continuum surfaces corresponding to the grand canonical ensemble scaling limit. Secondly, much remains to be established for the Ising+CT annealed model. A proof that it magnetizes at finite coupling, as seems almost certain from numerical work, would be significant progress; in the infinite graph ensemble, this would involve establishing how the measure differs from ρ . Assuming the model does have a continuous phase transition, it would be interesting to establish rigorously whether the critical exponents are shifted from the Onsager values.

正如我们在本文中所见，关于因果动态三角剖分 (CDT) 及其与自由度额外耦合到几何的扩展模型，仍存在若干待解决的问题，我们在此将其梳理总结。首先，仅能计算期望值的巨正则系综，与测度占优图具有诸多共性的无限图系综，二者之间的关系尚不明确。在其中一个系综里自然定义的“可观测量”，与在另一个系综里自然定义的可观测量存在细微差异。这种关系可以通过构造对应巨正则系综标度极限的连续曲面极限分布来确立。其次，Ising+CT 退火模型仍有许多问题有待解决。数值研究结果几乎已经可以确定，该模型在有限耦合下会发生磁化，对此给出证明将是重大进展；而在无限图系综中，这需要明确该模型的测度与 ρ 有何区别。若假设该模型确实存在连续相变，那么严格证明临界指数是否偏离 Onsager 值将是一个有意思的研究方向。

Obviously, it is of interest to study the CDT model in higher dimensions. In three dimensions, some progress has been made (see, for example, [35]), and it has been proved that the number of different three-dimensional CT grows exponentially with the volume so the partition functions one would like to analyze do converge for a range of coupling constants [36]. In four dimensions, there are essentially only numerical results as described in other contributions to this book. An important outstanding problem is to prove an exponential bound on the number of four-dimensional CT as a function of volume. In [36], it is shown that such a bound would follow from an exponential bound on the number of unrestricted three-dimensional triangulations of the sphere, as a function of volume.

显然，研究更高维度下的 CDT 模型颇具价值。三维模型的研究已经取得了一些进展 (例如参见文献 [35])，目前已经证明，不同三维因果三角剖分 (CT) 的数量随体积呈指数增长，因此人们想要分析的配分函数在一定耦合常数范围内是收敛的 [36]。四维模型目前基本上只有本书其他章节所述的数值结果。一个重要的待解决问题是，证明四维 CT 的数量关于体积存在指数界。文献 [36] 指出，该指数界可由球面无约束三维三角剖分的数量关于体积的指数界推导得出。

Cross-References

交叉引用

CDT and Hořava-Lifshitz QG in Two Dimensions

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